Incremental linear approximation of large deformations in finite elasticity

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Abstract

A method of incremental linear approximation for solving boundary value problems of large deformation in finite elasticity is considered. Instead of solving the nonlinear problem in Lagrangian formulation, by assuming time steps small enough and the reference configuration updated at every step, we can linearize the constitutive equation and reduce it to linear boundary value problems to be solved successively with incremental boundary data.

Keywords. Lagrangian formulation, nonlinear elasticity, linear approximation, updated reference configuration, boundary value problem, pure shear.

1 Introduction

The constitutive equation of a solid body is usually expressed relative to a preferred reference configuration. The behaviors for large deformations are characterized by some nonlinear constitutive functions, which lead to a system of nonlinear partial different equations for the solution of boundary value problems. The problems are usually formulated in Lagrangian coordinates in the preferred reference configuration, and numerical methods involve solving nonlinear systems as well as other difficulties, such as the boundary conditions may involve the deformed state which depends on the solutions themselves. Alternatively, we are proposing a different method for solving boundary value problem of large deformation in a successive Lagrangian formulation of incremental linear approximation.

Roughly speaking, the method of incremental linear approximation is similar to the Euler method for solving differential equations, i.e., successively at each state, the tangent is calculated and projected to a neighboring state. In other words, the constitutive equations are calculated at each state which will be regarded as the reference configuration (updated Lagrangian formulation) for the next state, and assuming the deformation to the next state is small, the updated constitutive equations are linearized. In this manner, it becomes a linear problem just like the problems in linear elasticity from one state to the next state with incremental small deformations.

2 Updated reference configurations and linearized constitutive equations

Let κ_r be the preferred reference configuration of an elastic body $\mathcal{B}$, such that the constitutive equation with respect to κ_r is given by

$$T = \mathcal{F}_{\kappa_r}(F), \tag{1}$$ (2.1)

where $T(X,t)$ is the cauchy stress, $X \in \kappa_r(\mathcal{B})$, and $F(X,t)$ is the deformation gradient with respect to κ_r.

It is well-known that the constitutive model based on the linear constitutive equation, Hooke's law, does not satisfy the principle of material frame-indifference, and it can only be regarded as an approximation of some nonlinear model for small deformations (see [1, 2]). Therefore, in order to consider large deformations, the constitutive function $\mathcal{F}$ is generally a nonlinear function of the deformation F.

Let κ_{t_0} be the deformed configuration at time t_0. The deformation from κ_r to κ_{t_0} need not be small. Let $F_0 = F(X,t_0)$ and $T_0 = T(X,t_0)$ be the deformation gradient and the Cauchy stress of the body at time t_0 respectively.

Now, consider a small deformation from κ_{t_0} to the configuration κ_t at time $t > t_0$ such that $F = (1+H)F_0$, where the displacement gradient $H = \nabla_{\boldsymbol{X}} \boldsymbol{u}$ relative to κ_{t_0} is small, $|H| \ll 1$.

$$X \in \kappa_r(\mathcal{B}) \xrightarrow{\quad F_0(X) \quad} \boldsymbol{X} \in \kappa_{t_0}(\mathcal{B}) \xrightarrow{\quad 1 + H(\boldsymbol{X}) \quad} \boldsymbol{x} \in \kappa_t(\mathcal{B})$$

We can then linearize the constitutive equation (1) relative to the configuration κ_{t_0}, namely,

$$T = T_0 + \nabla_F \mathcal{F}(F_0)[F - F_0] = T_0 + \nabla_F \mathcal{F}(F_0)[HF_0],$$

or

$$T = T_0 + L(F_0)[H], \tag{2}$$ (T-cauchy)

where

$$L(F_0)[H] = \nabla_F \mathcal{F}(F_0)[HF_0] \tag{3}$$ (L-elast)

defines the elasticity tensor relative to the reference configuration κ_{t_0}.

Since the reference stress T_0 and the deformation gradient F_0 are given in the *updated* reference state κ_{t_0}, the elasticity tensor $L(F_0)$ is a constant fourth order tensor.

Furthermore, the (first) Piola-Kirchhoff stress tensor T_κ relative to the updated reference configuration κ_{t_0} is given by

$$\begin{aligned} T_{\kappa_{t_0}} &= \det(1+H)T(1+H)^{-T} = \det(1+H)\big(T_0 + L(F_0)[H]\big)(1+H)^{-T} \\ &= (1+\operatorname{tr} H)\big(T_0 + L(F_0)[H]\big)(1-H^T) + o(2) \\ &= T_0 + (\operatorname{tr} H)T_0 - T_0H^T + L(F_0)[H] + o(2), \end{aligned}$$

where $o(2)$ represents higher order terms in the small displacement gradient $|H|$. We can also define the elasticity tensor for the Piola-Kirchhoff stress by

$$L_{\kappa_{t_0}}(F_0)[H] = (\operatorname{tr} H)T_0 - T_0H^T + L(F_0)[H], \qquad (4) \text{ (L-Piola)}$$

and write

$$T_{\kappa_{t_0}} = T_0 + L_{\kappa_{t_0}}(F_0)[H]. \qquad (5) \text{ (T-Piola)}$$

Note that $L_{\kappa_{t_0}}$ is different from the elasticity tensor L defined in (3), unless the reference state is a natural state, i.e., $T_0 = 0$. In this case, there is no difference between the Cauchy stress and the Piola-Kirchhoff stress, because the difference in higher order terms are neglected in the linear theory. However, since the reference state is not necessary stress-free, the relation (4) is important for the updated Lagrangian formulation of the problem.

3 Incremental linear approximation

We can now consider the updated Lagrangian formulation of the boundary value problem.

Let $\Omega \subset I\!R^3$ be the region occupied by the body at the configuration $\kappa = \kappa_{t_0}$, $\partial\Omega = \Gamma_1 \cup \Gamma_2$, and $\boldsymbol{n}_\kappa$ be the exterior normal to $\partial\Omega$. Let $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$ be the displacement vector relative to the configuration κ. Consider the boundary value problem of an elastic body in equilibrium without external body force given by

$$\begin{cases} -\operatorname{Div} T_\kappa = 0 & \text{in} \quad \Omega \times (t_0, t], \\ T_\kappa \boldsymbol{n}_\kappa = \boldsymbol{f} & \text{on} \quad \Gamma_1 \times (t_0, t], \\ \boldsymbol{u} \cdot \boldsymbol{n}_\kappa = \boldsymbol{g} & \text{on} \quad \Gamma_2 \times (t_0, t], \end{cases} \qquad (6) \text{ (bvp)}$$

where the prescribed surface traction $\boldsymbol{f}$ and the displacement $\boldsymbol{g}$ are assumed to be time-dependent. In numerical solutions, one can take the time step small enough so that linearized constitutive equation will be applicable for the small *incremental* boundary values. The idea is similar to the theory of small deformations superposed on finite deformations (see [2, 3]) at every step.

In this manner, either we are interested in the evolution of solutions with gradually changing boundary conditions resulting in large deformation, or, we can treat the boundary values of finite elasticity as the final value of a successive small incremental boundary values at each time step.

Assuming that at t_0, the deformation gradient F_0 with respect to the preferred configuration κ_r and the Cauchy stress T_0 are known, and that $\Delta t = t - t_0$ is small enough, then from (4) and (5), we have

$$T_\kappa = T_0 + (\operatorname{tr} H)T_0 - T_0 H^T + L(F_0)[H] = T_0 + L_\kappa(F_0)[H].$$

Upon substitution into (6), we have the following problem:

$$\begin{cases} -\operatorname{Div}(L_\kappa(F_0)[H]) = \operatorname{Div} T_0 & \text{in} \quad \Omega \times (t_0, t], \\ (L_\kappa(F_0)[H])\, \boldsymbol{n}_\kappa = \boldsymbol{f} - T_0 \boldsymbol{n}_\kappa & \text{on} \quad \Gamma_1 \times (t_0, t], \\ \boldsymbol{u} \cdot \boldsymbol{n}_\kappa = \boldsymbol{g} & \text{on} \quad \Gamma_2 \times (t_0, t], \end{cases} \tag{7}$$ (BVP)

where $H = \nabla_{\boldsymbol{X}} \boldsymbol{u}$. This is the same problem to determine the displacement vector $\boldsymbol{u}(\boldsymbol{X}, t)$ as the boundary value problem in linear elasticity, except the additional force terms due to the reference stress on the right hand side.

After solving the problem (7), the reference configuration can be updated, and the deformation gradient and the reference stress can be calculated so that the Lagrangian formulation of the problem in the form (7) can proceed again from the updated reference configuration. This numerical procedure will be referred to as the *successive Lagrangian formulation of incremental linear approximation for large deformations*, or simply as the method of *Incremental Linear Approximation* (ILA).

4 Numerical example: pure shear

As an interesting example for applying the method of ILA, we consider the case of a pure shear of a square block, by applying tangential surface traction, the shear stress τ, on the surfaces of the block as shown in Fig 1.

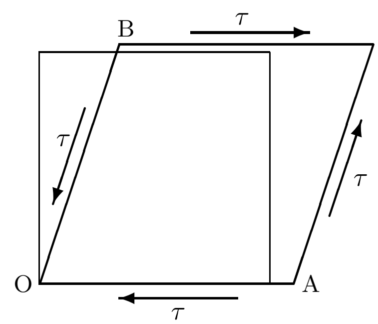

Figure 1: Pure shear

The pure shear is a homogeneous deformation and can easily solved analytically in finite elasticity (see for example, [1, 4]). There is an interesting universal relation associated with this problem for any elastic body, namely, the deformed state of the square block has the geometric form of an equilateral parallelogram, i.e., $\overline{OA} = \overline{OB}$ as shown in Fig. 1.

As an example, we shall consider a (compressible) Mooney-Rivlin material with the constitutive equation given by

$$T = \mathcal{F}_{\kappa_r}(F) = s_0 1 + s_1 B + s_2 B^{-1},$$

where $B = FF^T$ is the left Cauchy-Green strain tensor and the coefficients are constants. Moreover, we shall assume that the reference configuration κ_r is a natural state so that $s_0 + s_1 + s_2 = 0$.

From (3) and (4), after taking the gradient of $\mathcal{F}_{\kappa_r}(F)$ at F_0, we have

$$L(F_0)[H] = s_1(HB_0 + B_0H^T) - s_2(B_0^{-1}H + H^T B_0^{-1}),$$

and

$$L_{\kappa_{t_0}}(F_0)[H] = (\operatorname{tr} H)\, T_0 - T_0 H^T + s_1(HB_0 + B_0H^T) - s_2(B_0^{-1}H + H^T B_0^{-1}).$$

For the boundary conditions, we shall take the time-dependent shear stress $\tau = \alpha\, t$ for some small constant α. And for uniqueness of solution, the point O is held fixed and the line $\overline{OA}$ is in the horizontal direction as shown in Fig. 1.

We consider time step $t_n = n\Delta t$, so that the prescribed shear stress at time step t_n is $\tau = n\, \Delta\tau$ for $\Delta\tau = \alpha\, \Delta t$.

Following the method of ILA, at every step n, the reference stress T_0 and the deformation gradient F_0 are obtained from the previous step. Since the initial reference configuration κ_r is a natural state, for $n = 0$, we have $T_0 = 0$ and $F_0 = 0$.

The finite element method is used and the following data are given for numerical solutions: $s_1 = 1$, $s_2 = -0.2$, $\Delta\tau = 0.005$ and $\Omega = (0,1) \times (0,1)$.

The numerical results are shown in Fig. 2, in which the initial square and its deformed states at the applying shear stress $\tau = 40\,\Delta\tau$, $\tau = 80\,\Delta\tau$, and $\tau = 120\,\Delta\tau$ are shown. One can easily see that the geometric shapes of the deformed state are equilateral parallelograms as expected. Indeed, the errors for $n = 40$, $n = 80$, and $n = 120$ are respectively of 0.04%, 0.34% and 1.33%, by comparing the consecutive sides of the parallelogram, $(\overline{0A} - \overline{0B})/\overline{0A}$, using a mesh of 10×10 elements.

Remarks: For an amount of shear stress τ given, the solution for the deformation of the pure shear can be analytically determined once the constitutive equation is given. It can be determined numerically also by the ILA procedure as shown here by gradually increasing the shear stress until the given amount is reached.

However, if one try to solve the boundary value problem with the nonlinear constitutive equation directly, one might encounter a difficulty that the shear stresses have to be applied tangentially on the (slanted) surfaces of the deformed body, which is itself unknown before the problem is solved, in other words, the boundary conditions depend on the solution itself. On the other hand, in the

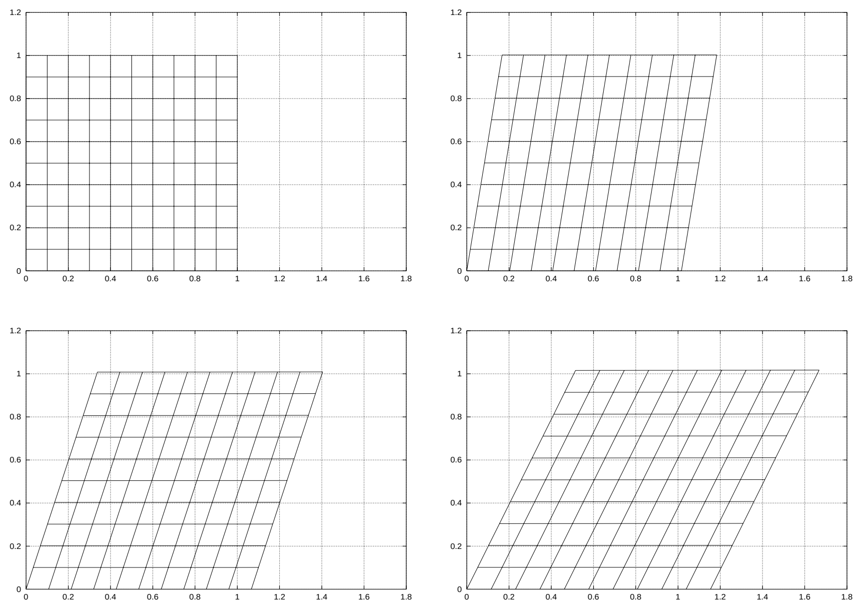

Figure 2: The square at ($n = 0$) and its subsequent deformed states at $n = 40$, $n = 80$, and $n = 120$

ILA procedure, it is of no concern at all, because in consecutive steps, it is a linear problem. Therefore, the difference between applying the same forces tangentially on the surfaces of the consecutive states is of higher orders which is insignificant in the linear approximation.

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