BEAM EQUATION WITH WEAK-INTERNAL DAMPING IN DOMAIN WITH MOVING BOUNDARY *

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Abstract

The small-amplitude motion of an elastic beam with internal damping is investigated in one-dimensional domain with moving boundary. Existence, uniqueness, asymptotic behavior, and numerical analysis of solutions are shown to exist to the mixed problem associated with the beam equation with fully clamped boundary conditions.

1 Introduction

The one-dimensional equation of motion of a thin beam with weak-internal damping undergoing cylindrical bending can be written as

$$\begin{aligned} u_{tt}(x,t) - \Big(\zeta_0 + \zeta_1 \int_\Omega |u_x(t)|^2 dx\Big) u_{xx}(x,t) \\ + u_{xxxx}(x,t) + \nu u_t(x,t) = 0 \quad \text{in} \quad Q, \end{aligned} \tag{1.1}$$

where u is the transverse displacement, x is the spatial coordinate in the longitudinal direction, t is the time. The aerodynamic damping term is denoted by ν, ζ_1 is the nonlinear stiffness of beam, ζ_0 is an in-plane tensile load, and (x,t) belongs to $Q = \Omega \times [0,\infty[$ with $\Omega = (0,1)$. All quantities are physically nondimensionalized, ν, ζ_1 are fixed positive and ζ_0 is not necessarily positive.

Our goal here is to inquire the mixed problem associated with equation (1.1) inside of a moving domain given by

$$\Omega_t = \big\{ x \in \mathbb{R} : \ \alpha(t) < x < \beta(t) \quad \text{and} \quad t \in [0,\infty[\ \big\},$$

$$Q_t = \big\{ (x,t) \in \mathbb{R}^2 : \ x \in \Omega_t \quad \text{and} \quad t \in [0,\infty[\ \big\},$$

$$\Sigma_t = \bigcup_{0<t<\infty} \big(\alpha(t) \times \beta(t), t\big).$$

where α and β are positive real-value functions.

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Equation (1.1) will be studied with the fully clamped boundary conditions, namely

$$u(\alpha(t),t) = u_x(\alpha(t),t) = 0 \quad \text{and} \quad u(\beta(t),t) = u_x(\beta(t),t) = 0. \tag{1.2}$$

Without the damping term, Cauchy's problem and mixed problem associated with equation (1.1), inside bounded and unbounded cylindrical domain have been studied by several authors. Existence and uniqueness of solutions were established, see for instance, Ball [1, 2], Dickey [7], and Medeiros [16].

Cauchy's problem associated with the equation (1.1) in an abstract framework on a Hilbert space H has also been studied by several authors, among them, Biler [3], Brito [4, 5] and Pereira [23] who aired result about existence, uniqueness and asymptotic stability of solutions.

Kirchhoff's equation with weak-internal damping can be obtained from equation (1.1) by omission of the term u_{xxxx}, e.g.,

$$u_{tt}(x,t) - \Big(\zeta_0 + \zeta_1 \int_\Omega |u_x(t)|^2 dx\Big) u_{xx}(x,t) + \nu u_t(x,t) = 0 \quad \text{in} \quad Q. \tag{1.3}$$

Equation (1.3) has been extensively studied by several authors in both $\{1, 2, \cdots, n\}$-dimensional cases and general mathematical model over a Hilbert space H. Local solutions and global solutions have been shown to exist in several physical-mathematical contexts. Among them, Kirchhoff [11], Matos & Pereira [15], Medeiros & Milla Miranda [18], Medeiros at al [17], Narashinham [20], Pohozaev [24], and a number of other interesting references cited in the previously mentioned papers, mainly in Medeiros at al [17].

Medeiros at al [17] introduced a new model associated with Kirchhoff's equation (1.3), which we denominate Medeiros-Kirchhoff's equation, namely

$$u_{tt}(x,t) - \Big(a(t) + b(t) \int_{\Omega_t} |u_x(t)|^2 dx\Big) u_{xx}(x,t) = 0 \quad \text{in} \quad Q_t, \tag{1.4}$$

where the coefficients $a(t) = \dfrac{\tau_0}{m} + \dfrac{k\big(\gamma(t) - \gamma_0\big)}{m\gamma_0}$ and $b(t) = \dfrac{k}{2m\gamma(t)}$ are defined in §1 of Medeiros at al [17]. The authors proved that the mixed problem associated with the equation (1.4) has local solutions. Besides, whether the equation (1.4) is considered over the action of a weak-internal damping $u_t(x,t)$, and the initial datum are bounded, then the total energy of the system is asymptotically stable. All these results were got on the hypothesis of that the domain Ω_t is increasing.

Considering equation (1.1) inside of a non-cylindrical domain Q_t and comparing with the equation (1.4) we deduce that ζ_0 and ζ_1 depend continuously on the length of domain Ω_t. Therefore, depend on the variable time t. But, to simplify our analysis we will assume that ζ_0 and ζ_1 are constants.

The strategy to obtain existence of a solution to the mixed problem associated with equation (1.1) in a non-cylindrical domain is to transform such mixed problem in another, defined over a cylindrical domain whose sections are not time dependent. This technique that transforms the equation from moving boundary in fixing boundary was initially utilized in the equation (1.3) by Ferrel & Medeiros [8] and proceed by Medeiros at al [17].

Organization of the work. In section 2 we will investigate the global existence and uniqueness of strong and weak solutions of the mixed problem for the equation (1.1) with fully clamped boundary conditions (1.2) on moving ends. Section 3 will be devoted to prove the asymptotic behavior of the total energy associated with the weak solutions inside of a non-cylindrical domain. Section 4 will contain the numerical simulations of the solutions, which are obtained by using both finite element and finite difference methods.

2 Existence and uniqueness of strong and weak solutions

Our goal in this section is to show existence of strong and weak solutions, as well as their uniqueness for the mixed problem for (1.1) with boundary conditions (1.2). That is,

$$\begin{aligned} u_{tt}(x,t) - \Big(\zeta_0 + \zeta_1 \int_{\Omega_t} |u_x(t)|^2 dx\Big) u_{xx}(x,t) \\ + u_{xxxx}(x,t) + \nu u_t(x,t) = 0 \quad \text{in} \quad Q_t, \end{aligned} \tag{2.1}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega_0, \tag{2.2}$$

$$u(x,t) = u_x(x,t) = 0 \quad \text{on} \quad \Sigma_t. \tag{2.3}$$

Now we will transform the mixed problem (2.1)-(2.3) in another initial data-and boundary-value problem over a cylindrical domain. That is, we will use the suitable change of variable established by the following function

$$\begin{aligned} \mathcal{T} \,:\, Q_t \to Q = (0,1) \times [0,\infty[\\ (x,t) \mapsto (y,t) = \Big(\frac{x-\alpha(t)}{\gamma(t)}\,,\, t\Big), \end{aligned} \tag{2.4}$$

where $\gamma(t) = \beta(t) - \alpha(t)$ for all $t \geq 0$. The functions α, β and γ satisfy the following hyphotesis

$$\alpha,\ \beta \in W^{3,1}\,([0,\ \infty[;\mathbb{R}) \ \text{ and } \ 0 < \gamma_0 = \gamma(0) \leq \gamma(t). \tag{2.5}$$

Using the function (2.4) and denoting by

$$v(y,t) = \big(u \circ \mathcal{T}^{-1}\big)\,(y,t) = u(x,t)$$

we have the following identity

$$u_t = v_t - \frac{1}{\gamma}\,(\alpha' + y\gamma')\,v_y, \tag{2.6}$$

$$u_{tt} = \frac{1}{\gamma^2}\left\{2\gamma'(\alpha' + y\gamma') - \gamma(\alpha'' + y\gamma'')\right\} v_y - \frac{2}{\gamma}(\alpha' + y\gamma')v_{yt} + \frac{1}{\gamma^2}(\alpha' + y\gamma')^2 v_{yy} + v_{tt}, \tag{2.7}$$

$$\partial_x{}^k u = \frac{1}{\gamma^k}\partial_y{}^k v \quad \text{for all} \quad k \in \mathbb{N}. \tag{2.8}$$

Taking into account (2.6)-(2.8) into (2.1)-(2.3) yields the following mixed problem with variable coefficients

$$v_{tt}(y,t) + a_1(y,t)v_y(y,t) + a_2(y,t)v_{yt}(y,t) + a_3(y,t)v_{yy}(y,t) + b_1(t)\Big(\int_\Omega |v_y(t)|^2 dy\Big)v_{yy}(y,t) + b_2(t)v_{yyyy}(y,t) + \nu v_t(y,t) = 0 \quad \text{in} \quad Q, \tag{2.9}$$

$$v(y,0) = v_0(y), \quad v_t(y,0) = v_1(y) \quad \text{with} \quad 0 < y < 1, \tag{2.10}$$

$$v(0,t) = v_y(0,t) = 0 \quad \text{for all} \quad t \geq 0, \qquad v(1,t) = v_y(1,t) = 0 \quad \text{for all} \quad t \geq 0, \tag{2.11}$$

where the coefficients of equation (2.9) are given by

$$a_1(y,t) = \tfrac{1}{\gamma^2}\left\{(2\gamma' - \nu\gamma)(\alpha' + y\gamma') - \gamma(\alpha'' + y\gamma'')\right\},$$
$$a_2(y,t) = -\tfrac{2}{\gamma}(\alpha' + y\gamma'), \quad a_3(y,t) = \tfrac{1}{\gamma^2}\left\{(\alpha' + y\gamma')^2 - \zeta_0\right\}, \tag{2.12}$$
$$b_1(t) = -\tfrac{\zeta_1}{\gamma^3}, \quad b_2(t) = \tfrac{1}{\gamma^4}.$$

Thanks to function $\mathcal{T}$, u is a solution of the mixed problem (2.1)-(2.3) if, and only if v is a solution of the mixed problem (2.9)-(2.11). Thus, we have that the two following theorems are equivalent

Theorem 2.1 (STRONG SOLUTION) *The nonlinear mixed problem (2.1)-(2.3) has a unique solution $u : [0,\infty[\to \mathbb{R}$ provided that α and β satisfy the hypothesis (2.5), the initial displacement u_0 belongs to $H_0^2(\Omega_0) \cap H^4(\Omega_0)$, and the initial velocity u_1 belongs to $H_0^2(\Omega_0)$. The solution u lies in*

$$u \in L^\infty\left(0,\infty; H_0^2(\Omega_t) \cap H^4(\Omega_t)\right), \tag{2.13}$$

$$u_t \in L^\infty\left(0,\infty; H_0^2(\Omega_t)\right), \tag{2.14}$$

$$u_{tt} \in L^\infty\left(0,\infty; L^2(\Omega_t)\right), \tag{2.15}$$

and the equation (2.1) is verified in the sense of $L^\infty\left(0,\infty; L^2(\Omega_t)\right)$.

Theorem 2.2 *Let α and β satisfying the hypotheses of the Theorem 2.1, and*

$$\frac{|\gamma'(t)|}{\gamma(t)} \leq \frac{1}{8} \quad \textit{for all} \quad t \geq 0. \tag{2.16}$$

If v_0 belongs to $H_0^2(\Omega)\cap H^4(\Omega)$ and v_1 belongs to $H_0^2(\Omega)$, the unique solution v of the mixed problem (2.9)-(2.11) lies in

$$v \in L^\infty\left(0,\infty; H_0^2(\Omega)\cap H^4(\Omega)\right), \tag{2.17}$$

$$v_t \in L^\infty\left(0,\infty; H_0^2(\Omega)\right), \tag{2.18}$$

$$v_{tt} \in L^\infty(0,\infty; L^2(\Omega)), \tag{2.19}$$

and the equation (2.9) is verified in the sense of $L^\infty(0,\infty;L^2(\Omega))$.

Proof of Theorem 2.2 - Existence. The Galerkin's method is the natural technique to solve the problem (2.9)-(2.11) in view of the coefficients of equation (2.9) are variable. Thus, let us consider the vector set $(w_j)_{j\in\mathbb{N}}$ complete and orthonormalized in $L^2(\Omega)$ with $w_j \in H_0^2(\Omega)$. We will denote by $V_m = [w_1, w_2, w_3, \cdots, w_m]$ the subspace of $H_0^2(\Omega)\cap H^4(\Omega)$ spanned by the first m vectors of $(w_j)_{j\in\mathbb{N}}$. The vectors $\psi_m(t)$ of V_m are represented by

$$\psi_m(t) = \sum_{i=1}^{m} g_{im}(t) w_i(y), \tag{2.20}$$

and we will assume that the initial datas v_0^m and v_1^m in V_m satisfy

$$\begin{aligned} &v^m(y,0) = v_0^m \longrightarrow v_0 \quad \text{strongly} \quad \text{in} \quad H_0^2(\Omega)\cap H^4(\Omega),\\ &v_t^m(y,0) = v_1^m \longrightarrow v_1 \quad \text{strongly} \quad \text{in} \quad H_0^2(\Omega). \end{aligned} \tag{2.21}$$

As $v^m(t)$ belongs to V_m, the system (2.9)-(2.11) on V_m becomes the following approximate system of ordinary differential equations

$$\begin{aligned} &(v_{tt}^m(t),\phi) + \left(a_1(t)v_y^m(t),\phi\right) + \left(a_2(t)v_{yt}^m(t),\phi\right) +\\ &\left(a_3(t)v_{yy}^m(t),\phi\right) + b_1(t)\Big(\int_\Omega \left|v_y^m(t)\right|^2 dy\Big)\left(v_{yy}^m(t),\phi\right) +\\ &b_2(t)\left(v_{yy}^m(t),\phi_{yy}\right) + \nu\left(v_t^m(t),\phi\right) = 0,\\ &v^m(y,0) = v_0^m \quad \text{and} \quad v_t^m(y,0) = v_1^m, \end{aligned} \tag{2.22}$$

where henceforth $(\cdot,\cdot)$ denotes the inner product on $L^2(\Omega)$.

Suppose that (2.21) holds, system (2.22) has a local solution $v^m(t)$ over the interval $[0,t_m[$. This interval will be extended to any interval $[0,\infty[$ thanks to *estimate I* that will be determinate below. For the sake of simplicity, we will write, throughout this section, v instead of v^m.

Our next task is to determine estimates on v sufficients to take the limit in the equation $(2.22)_1$.

Estimate I. Substituting ϕ by $2v_t(t)$ into $(2.22)_1$, denoting by $\widehat{M}$ the function $\widehat{M}(\xi)=\int_0^\xi M(\tau)d\tau$, and taking into account the following relations

$$b_3(t):=\sup_{y\in[0,1]}|a_1(y,t)|_{\mathbb{R}}\leq\frac{1}{\gamma_0^2}\left|2\gamma'(t)-\nu\gamma(t)\right|\max\Big\{|\alpha'(t)|_{\mathbb{R}},|\beta'(t)|_{\mathbb{R}}\Big\}+\frac{1}{\gamma_0}\max\Big\{|\alpha''(t)|_{\mathbb{R}},|\beta''(t)|_{\mathbb{R}}\Big\},$$

$$b_4(t):=\sup_{y\in[0,1]}\left|\frac{1}{\gamma^2(t)}\Big(\alpha'(t)+y\gamma'(t)\Big)^2\right|_{\mathbb{R}}\leq\frac{1}{\gamma_0^2}\left[\max\left\{|\alpha'(t)|_{\mathbb{R}},\ |\beta'(t)|_{\mathbb{R}}\right\}\right]^2,$$

$$2(a_2(t)v_{yt}(t),v_t(t))=\frac{2\gamma'(t)}{\gamma(t)}\left|v_t(t)\right|^2,$$

we get

$$\frac{d}{dt}E_1(t)+2\Big(\frac{\gamma'(t)}{\gamma(t)}+\nu\Big)|v_t(t)|^2\leq b_1'(t)\widehat{M}\left(|v_y(t)|^2\right)+|b_2'(t)|\left|v_{yy}(t)\right|^2+2b_3(t)\left|v_y(t)\right|\left|v_t(t)\right|+2b_4(t)\left|v_{yy}(t)\right|\left|v_t(t)\right|, \tag{2.23}$$

where

$$E_1(t)=\left|v_t(t)\right|^2-b_1(t)\widehat{M}\left(|v_y(t)|^2\right)+b_2(t)\left|v_{yy}(t)\right|^2,$$

and $|\cdot|^2$ denotes the norm on $L^2(\Omega)$.

The last two terms of right-hand side of preceding inequalities satisfy

(a) $$2b_3(t)\left|v_y(t)\right|\left|v_t(t)\right|\leq\frac{b_3^2(t)}{\nu}\left|v_{yy}(t)\right|^2+\nu\left|v_t(t)\right|^2,$$

(b) $$2b_4(t)\left|v_{yy}(t)\right|\left|v_t(t)\right|\leq\frac{b_4^2(t)}{\nu}\left|v_{yy}(t)\right|^2+\nu\left|v_t(t)\right|^2.$$

Taking into account (a) and (b) into (2.23) yields

$$\frac{d}{dt}E_1(t)\leq2\frac{|\gamma'(t)|}{\gamma(t)}|v_t(t)|^2+b_1'(t)\widehat{M}\left(|v_y(t)|^2\right)+b_5(t)\left|v_{yy}(t)\right|^2, \tag{2.24}$$

where

$$b_5(t)=|b_2'(t)|+\frac{1}{\nu}\big(b_3^2(t)+b_4^2(t)\big).$$

Integrating (2.24) from 0 to $t\leq T$ yields

$$E_1(t)\leq E_1(0)+\int_0^t b_6(s)\left\{|v_s(s)|^2+\widehat{M}\left(|v_y(s)|^2\right)+|v_{yy}(s)|^2\right\}ds, \tag{2.25}$$

where $b_6(t) = 2\frac{|\gamma'(t)|}{\gamma(t)} + |b_1'(t)| + b_5(t) \leq |\gamma'(t)| \left(\frac{2}{\gamma_0} + \frac{3\zeta_1}{\gamma_0^4} + \frac{4}{\gamma_0^5}\right) + \frac{1}{\nu}\left(b_3^2(t) + b_4^2(t)\right)$. The hypothesis (2.5) implies that $b_6 \in L^1(0,\infty)$, and $E_1(0)$ is bounded. Thus, applying the Gronwall's inequality in (2.25) yields

$$\begin{aligned} &(v^m) \quad \text{is bounded in} \quad L^\infty\left(0,\infty; H_0^2(\Omega)\right), \\ &(v_t^m) \quad \text{is bounded in} \quad L^\infty\left(0,\infty; L^2(\Omega)\right). \end{aligned} \tag{2.26}$$

Estimate II. Differentiating the equation $(2.22)_1$ with respect to t, replacing ϕ by $2v_{tt}(t)$, and using the following identities

- $$2\left(a_2(t)v_{ytt}(t), v_{tt}(t)\right) = \frac{2\gamma'(t)}{\gamma(t)}\left|v_{tt}(t)\right|^2,$$
- $$2b_2'(t)\left(v_{yy}(t), v_{yytt}(t)\right) = 2b_2'(t)\frac{d}{dt}\left(v_{yy}(t), v_{yyt}(t)\right) - 2b_2'(t)\left|v_{yyt}(t)\right|^2,$$

we get

$$\begin{aligned} &\frac{d}{dt}\left|v_{tt}(t)\right|^2 + 2b_2'(t)\frac{d}{dt}\left(v_{yy}(t), v_{yyt}(t)\right) + b_2(t)\frac{d}{dt}\left|v_{yyt}(t)\right|^2 + \frac{2\nu\gamma'(t)}{\gamma(t)}\left|v_{tt}(t)\right|^2 = \\ &-2\left(a_{1t}(t)v_y(t), v_{tt}(t)\right) - 2\left(a_1(t)v_{yt}(t), v_{tt}(t)\right) - 2\left(a_{2t}(t)v_{yt}(t), v_{tt}(t)\right) - \\ &2\left(a_{3t}(t)v_{yy}(t), v_{tt}(t)\right) - 2b_1(t)\Big(\int_\Omega \left|v_y(t)\right|^2 dy\Big)\left(v_{yyt}(t), v_{tt}(t)\right) - \\ &2\left(a_3(t)v_{yyt}(t), v_{tt}(t)\right) - 2b_1'(t)\Big(\int_\Omega \left|v_y(t)\right|^2 dy\Big)\left(v_{yy}(t), v_{tt}(t)\right) + \\ &2b_2'(t)\left|v_{yyt}(t)\right|^2 - 2b_1(t)\Big(\int_\Omega v_y(t)v_{yt}(t)dy\Big)\left(v_{yy}(t), v_{tt}(t)\right). \end{aligned}$$

Denoting by E_2 the function

$$E_2(t) = \left|v_{tt}(t)\right|^2 + 2b_2'(t)\left(v_{yy}(t), v_{yyt}(t)\right) + b_2(t)\left|v_{yyt}(t)\right|^2$$

we can write

$$\begin{aligned} &\frac{d}{dt}E_2(t) + \frac{2\nu\gamma'(t)}{\gamma(t)}\left|v_{tt}(t)\right|^2 = \\ &2b_2''(t)\left(v_{yy}(t), v_{yyt}(t)\right) + 3b_2'(t)\left|v_{yyt}(t)\right|^2 - \\ &2\left(a_{1t}(t)v_y(t), v_{tt}(t)\right) - 2\left(a_1(t)v_{yt}(t), v_{tt}(t)\right) - \\ &2\left(a_{2t}(t)v_{yt}(t), v_{tt}(t)\right) - 2\left(a_{3t}(t)v_{yy}(t), v_{tt}(t)\right) - \\ &2\left(a_3(t)v_{yyt}(t), v_{tt}(t)\right) - 2b_1(t)\Big(\int_\Omega v_y(t)v_{yt}(t)dy\Big)\left(v_{yy}(t), v_{tt}(t)\right) - \\ &2b_1'(t)\Big(\int_\Omega \left|v_y(t)\right|^2 dy\Big)\left(v_{yy}(t), v_{tt}(t)\right) - 2b_1(t)\Big(\int_\Omega \left|v_y(t)\right|^2 dy\Big)\left(v_{yyt}(t), v_{tt}(t)\right). \end{aligned}$$

Hence, proceeding as in the *estimate I*, using (2.26), the hypothesis (2.5), and usual inequalities, we obtain after some simplification that

$$\frac{d}{dt}E_2(t) \leq b_7(t) + \frac{\nu\,|\gamma'(t)|}{\gamma(t)}\,|v_{tt}(t)|^2 + b_8(t)\,|v_{yyt}(t)|^2\,, \tag{2.27}$$

where $b_7(t)$ and $b_8(t)$ are L^1- functions defined for all $t \geq 0$.

Defining by E_3 the function $E_3(t) = E_1(t) + E_2(t)$ and using (2.24), (2.27) yields

$$\frac{d}{dt}E_3(t) \leq b_7(t) + b_9(t)\Big\{|v_t(t)|^2 + \widehat{M}\left(|v_y(t)|^2\right) + |v_{yy}(t)|^2 + |v_{tt}(t)|^2 + |v_{yyt}(t)|^2\Big\},$$

where $b_9(t) = (2+\nu)\frac{\gamma'(t)}{\gamma(t)} + b_1'(t) + b_5(t) + b_8(t)$. Integrating the precedent inequality from 0 to $t \leq T$ and observing that $v_{tt}(0)$ is bounded in $L^2(\Omega)$ we have, thanks to Gronwall's inequality, that

$$E_3(t) \leq C \quad \text{for all} \quad t \geq 0. \tag{2.28}$$

By usual inequality we get the following inequalities

$$\begin{aligned} |2b_2'(t)\,(v_{yy}(t), v_{yyt}(t))| &\leq \frac{8|\gamma'(t)|}{\gamma^5(t)}\,|v_{yy}(t)|\,|v_{yyt}(t)| \leq \\ \frac{8|\gamma'(t)|}{\gamma(t)}\Big(\frac{1}{2\gamma^4(t)}\,|v_{yy}(t)|^2 &+ \frac{1}{2\gamma^4(t)}\,|v_{yyt}(t)|^2\Big) \leq \\ \frac{1}{2}b_2(t)\,|v_{yy}(t)|^2 &+ \frac{1}{2}b_2(t)\,|v_{yyt}(t)|^2\,, \end{aligned} \tag{2.29}$$

where we have used in the last inequality above the hypothesis (2.16). From definition of E_3 and (2.29) we have

$$\begin{aligned} |v_t(t)|^2 - b_1(t)\widehat{M}\left(|v_y(t)|^2\right) + |v_{tt}(t)|^2 + \\ \frac{1}{2}b_2(t)\,|v_{yy}(t)|^2 + \frac{1}{2}b_2(t)\,|v_{yyt}(t)|^2 \leq E_3(t) \quad \text{for all} \quad t \geq 0. \end{aligned} \tag{2.30}$$

Thus, from (2.28) and (2.30) we obtain

$$\begin{aligned} (v_{tt}^m) &\quad \text{is bounded in} \quad L^\infty\left(0,\infty; L^2(\Omega)\right), \\ (v_t^m) &\quad \text{is bounded in} \quad L^\infty\left(0,\infty; H_0^2(\Omega)\right). \end{aligned} \tag{2.31}$$

Limit of the approximate solutions. From estimates obtained in (2.26) and (2.31) we can take the limit of the nonlinear system (2.22). In fact, from (2.26) we have, in particular, that the sequences (v^m) and (v_t^m) are bounded in $L^2\big(0,T;H_0^1(\Omega)\big)$ and $L^2\big(0,T;L^2(\Omega)\big)$, respectively. Thus, by compact injection of $H_0^1(\Omega\times]0,T[)$ into $L^2(\Omega\times]0,T[)$ it follows by Lions-Aubin's theorem that there exists a subsequence of (v^m), which we denote as the original sequence, such that

$$v^m \longrightarrow v \quad \text{strong in} \quad L^2\big(0,T;H_0^1(\Omega)\big) \quad \text{as} \quad m\longrightarrow\infty. \tag{2.32}$$

We still can obtain from (2.26) and (2.31) the following convergence

$$\begin{aligned} v^m &\longrightarrow v \quad \text{weak star in} \quad L^\infty\big(0,\infty;H_0^2(\Omega)\big),\\ v_t^m &\longrightarrow v_t \quad \text{weak star in} \quad L^\infty\big(0,\infty;H_0^2(\Omega)\big),\\ v_{tt}^m &\longrightarrow v_{tt} \quad \text{weak star in} \quad L^\infty\big(0,\infty;L^2(\Omega)\big). \end{aligned} \tag{2.33}$$

Using (2.32) and $(2.33)_1$ we can take the limit in the nonlinear terms, i. e.,

$$\Big(\int_\Omega |v_y^m|^2\,dy\Big)v_{yy}^m \longrightarrow \Big(\int_\Omega |v_y|^2\,dy\Big)v_{yy}, \tag{2.34}$$

weak star in $L^\infty\big(0,\infty;L^2(\Omega)\big)$. Taking into account $(2.33)_{2,\,3}$ and (2.34) into $(2.22)_1$, there exists a function v defined over $\Omega\times[0,\infty[$ with value in $\mathbb{R}$ satisfying (2.9). The statement (2.18) and (2.19) are given by convergence $(2.33)_{2,\,3}$. The regularity (2.17) is guaranteed by $(2.33)_1$ and by results on elliptic regularity (see, for instance, Medeiros & Milla Miranda [19] or Nirenberg [22]) in view of

$$-v_{yyyy} = g \in L^\infty(0,\infty;L^2(\Omega)),$$

where $\quad g=\gamma^4\Big\{v''+a_1v_y+a_2v_{yt}+a_3v_{yy}+b_1\Big(\int_\Omega|v_y|^2dy\Big)v_{yy}+\nu v_t\Big\}.$

Hence, we conclude that equation (2.9) is given in the sense of $L^\infty(0,\infty;L^2(\Omega))$.

As consequence of $(2.33)_1$-$(2.33)_3$, the functions v and v_t are continuous (see, for example, L ions & Magenes [14], Vol. I, Cap. 1). Therefore, the initial conditions (2.10) are well defined.

Uniqueness. Let $w=v-\tilde v$ be, where v and $\tilde v$ are two solutions of the equation (2.9)-(2.11) over $[0,\infty[$. For all $\phi\in H_0^2(\Omega)$, and $t\geq 0$ the function w satisfies:

$$\begin{aligned} &(w_{tt}(t),\phi)+(a_1(t)w_y(t),\phi)+(a_2(t)w_{yt}(t),\phi)+\\ &(a_3(t)w_{yy}(t),\phi)+b_2(t)\,(w_{yy}(t),\phi_{yy})+\nu\,(w_t(t),\phi)+\\ &b_1(t)\Big(\int_\Omega |v_y(t)+\tilde v_y(t)|\,|w_y(t)|\,dy\Big)\,(w_{yy}(t),\phi)=0,\\ &w(y,0)=w_t(y,0)=0 \quad \text{for all} \quad y\in]0,1[,\\ &w(0,t)=\ w(1,t)\ =0 \quad \text{for all} \quad t\geq 0,\\ &w_x(0,t)=w_x(1,t)=0 \quad \text{for all} \quad t\geq 0. \end{aligned}$$

Replacing ϕ by w_t in the preceding system we obtain

$$\frac{1}{2}\frac{d}{dt}\left|w_t(t)\right|^2+\frac{1}{2}b_2(t)\frac{d}{dt}\left|w_{yy}(t)\right|^2+\nu\left|w_t(t)\right|^2=-$$

$$(a_1(t)w_y(t),w_t(t))-(a_2(t)w_{yt}(t),w_t(t))-(a_3(t)w_{yy}(t),w_t(t))-$$

$$b_1(t)\Big(\int_\Omega |v_y(t)+\tilde{v}_y(t)|\,|w_y(t)|\,dy\Big)\,(w_{yy}(t),w_t(t))\,.$$

By using the estimate (2.17), identity

$$-\left(a_2(t)w_{yt}(t),w_t(t)\right)=-\frac{\gamma'(t)}{\gamma(t)}\left|w_t(t)\right|^2$$

and usual inequalities in right-hand side of preceding identity, yields

$$\frac{1}{2}\frac{d}{dt}\left\{\left|w_t(t)\right|^2+b_2(t)\left|w_{yy}(t)\right|^2\right\}\le b_{10}(t)\left\{\left|w_t(t)\right|^2+\left|w_{yy}(t)\right|^2\right\},$$

where $b_{10}(t)$ is a positive $L^1(0,\infty)-$ function, thanks to hypothesis (2.5). Hence, integrating from 0 to $t\le T$ and the initial conditions yields

$$\frac{1}{2}\left|w_t(t)\right|^2+\frac{1}{2}b_2(t)\left|w_{yy}(t)\right|^2\le\int_0^t b_{10}(s)\left\{\left|w_s(s)\right|^2+\left|w_{yy}(s)\right|^2\right\}ds.$$

From this and Gronwall's inequality we obtain $w=0$ for all $t\in[0,+\infty[$. Thus, the demonstration of Theorem 2.2 is concluded ■

As was said before, the demonstration of Theorem 2.1 is established by function $\mathcal{T}$ defined in (2.4) ■

Our goal now is to find a solution for the system (2.1)-(2.3) with initial conditions in the class

$$u_0\in H_0^2(\Omega_0)\quad\text{and}\quad u_1\in L^2(\Omega_0).$$

Thus, we have

Theorem 2.3 (WEAK SOLUTION) *Let α and β satisfying the hypotheses of the Theorem 2.1 and the hypothesis (2.16) . If u_0 belongs to $H_0^2(\Omega_0)$ and u_1 belongs to $L^2(\Omega_0)$, then the unique solution u of the mixed problem (2.1)-(2.3) lies in*

$$u\in L^\infty\left(0,\infty;H_0^2(\Omega_t)\right), \tag{2.35}$$

$$u_t\in L^\infty\left(0,\infty;L^2(\Omega_t)\right), \tag{2.36}$$

$$u_{tt}\in L^\infty(0,\infty;H^{-2}\left(\Omega_t\right)), \tag{2.37}$$

and the equation (2.1) is verified in the sense of $L^\infty(0,\infty;H^{-2}(\Omega_t))$.

The equivalent theorem in the cylindrical domain is given by

Theorem 2.4 *Let α and β under the hypotheses of the Theorem 2.1. If v_0 belongs to $H_0^2(\Omega)$ and v_1 belongs to $L^2(\Omega)$, then the unique solution v of the mixed problem (2.9)-(2.11) lies in*

$$v \in L^\infty\left(0,\infty; H_0^2(\Omega)\right), \tag{2.38}$$

$$v_t \in L^\infty\left(0,\infty; L^2(\Omega)\right), \tag{2.39}$$

$$v_{tt} \in L^\infty(0,\infty; H^{-2}(\Omega)), \tag{2.40}$$

and the equation (2.9) is verified in the sense of $L^\infty(0,\infty;H^{-2}(\Omega))$.

Proof of Theorem 2.4 - Existence. The demonstration of existence of a weak solution is made by using the same method utilized to get the strong solution. However, we now consider the basis $(w_j)_{j\in\mathbb{N}}$ in the space $H_0^2(\Omega)$, and the initial conditions satisfying

$$v_0^m \longrightarrow v_0 \quad \text{strongly} \quad \text{in} \quad H_0^2(\Omega), \tag{2.41}$$

$$v_1^m \longrightarrow v_1 \quad \text{strongly} \quad \text{in} \quad L^2(\Omega). \tag{2.42}$$

Using the same procedure used to get *estimate I* of the Theorem 2.2 we get

$$\begin{aligned}
&|v_t(t)|^2 + b_1(t)\widehat{M}\left(|v_y(t)|^2\right) + b_2(t)|u_{yy}(t)|^2 \leq \\
&|v_1|^2 + b_1(0)\widehat{M}\left(|v_y(0)|^2\right) + b_2(0)|u_{yy}(0)|^2 + \\
&\int_0^t b_6(s)\left\{|v_s(t)|^2 + \widehat{M}\left(|v_y(s)|^2\right) + |u_{yy}(s)|^2\right\} ds,
\end{aligned} \tag{2.43}$$

for all $t \geq 0$. Hence, and Gronwall's inequality we have that the left-hand side of (2.43) is bounded. Thus, there exists a subsequence of $(v^m)_{m\in\mathbb{N}}$, which we denote as the original sequence, and a function $v : \Omega\times]0,\infty[\to \mathbb{R}$ such that

$$v^m \longrightarrow v \quad \text{weak star in} \quad L^\infty\left(0,\infty; H_0^2(\Omega)\right), \tag{2.44}$$

$$v_t^m \longrightarrow v_t \quad \text{weak star in} \quad L^\infty\left(0,\infty; L^2(\Omega)\right). \tag{2.45}$$

From $(2.22)_1$ we can write, for all $\phi \in H_0^2(\Omega)$, and for all $\psi \in \mathcal{D}(0,\infty)$ that

$$\begin{aligned}
&\int_0^\infty (v_{tt}^m(t),\phi)\,\psi(t)dt + \int_0^\infty \left(a_1(t)v_y^m(t),\phi\right)\psi(t)dt + \\
&\int_0^\infty \left(a_2(t)v_{yt}^m(t),\phi\right)\psi(t)dt + \int_0^\infty \left(a_3(t)v_{yy}^m(t),\phi\right)\psi(t)dt + \\
&\int_0^\infty b_2(t)\left(v_{yy}^m(t),\phi_{yy}\right)\psi(t)dt + \nu\int_0^\infty (v_t^m(t),\phi)\,\psi(t)dt + \\
&\left(\int_\Omega |v_y^m(t)|dy\right)\int_0^\infty b_1(t)\left(v_y^m(t),\phi_y\right)\psi(t)dt = 0.
\end{aligned}$$

Hence, by formal integration by parts and observing that

$$\big(a_2(t)v_{yt}^m(t),\phi\big) = -\left(a_2(t)v_t^m(t),\phi_y\right) + \frac{2\gamma'(t)}{\gamma(t)}\left(v_t^m(t),\phi\right), \tag{2.46}$$

we have

$$-\int_0^\infty \left(v_t^m(t),\phi\right)\psi'(t)dt + \int_0^\infty \big(a_1(t)v_y^m(t),\phi\big)\,\psi(t)dt-$$
$$\int_0^\infty \left(a_2(t)v_t^m(t),\phi_y\right)\psi(t)dt + \int_0^\infty \big(a_3(t)v_{yy}^m(t),\phi\big)\,\psi(t)dt+$$
$$\int_0^\infty b_2(t)\,\big(v_{yy}^m(t),\phi_{yy}\big)\,\psi(t)dt + \int_0^\infty \Big(\nu + \frac{2\gamma'(t)}{\gamma(t)}\Big)\left(v_t^m(t),\phi\right)\psi(t)dt+$$
$$\Big(\int_\Omega |v_y^m(t)|dy\Big)\int_0^\infty b_1(t)\,\big(v_y^m(t),\phi_y\big)\,\psi(t)dt = 0.$$

Hence, from (2.44), (2.45), (2.46), and as the operator ∂_y^4 is an isomorphism from $H_0^2(\Omega)$ into $H^{-2}(\Omega)$ we can conclude that the function v satisfies

$$\begin{aligned} & v_{tt} + a_1 v_y + a_2 v_{yt} + a_3 v_{yy} + \\ & b_1\Big(\int_\Omega |v_y|^2 dy\Big)v_{yy} + b_2 v_{yyyy} + \nu v_t = 0, \end{aligned} \tag{2.47}$$

in the sense of $L^2\left(0,\infty;H^{-2}(\Omega)\right)$. Therefore, the statement (2.38)-(2.40) are consequence of the convergence (2.44), (2.45), and of the equation (2.47). Thanks to (2.38)-(2.40) we have that the functions v and v_t are continuous. Thus, the initial conditions (2.10) are verified.

Uniqueness. From (2.39) and (2.40) the duality $\langle u_{tt}, u_t\rangle_{H^{-2}(\Omega)\times L^2(\Omega)}$ does not make sense. Thus, it is necessary to utilize the regularization's method of Ladyzhenskaya-Visik [12], see also Lions [13] (Theorem 1.2, pp. 14), and we proceed such as in the demonstration of uniqueness of Theorem 2.2 . Therefore, we have that the proof of the Theorem 2.4 is concluded. Finally, we obtain by using the function $\mathcal{T}$, all the statement of the Theorem 2.3 ■

3 Asymptotic behavior

In this section our goal is to show that the total energy of the system (2.1)-(2.3) decay exponentially when the time t goes to $+\infty$.

The structure of non-cylindrical domain Q_t has a particular importance in the prove of the asymptotic stability of the energy. This way, we assume that the non-cylindrical Q_t is increasing. It means that, the functions α and β satisfy the following properties

$$\alpha,\ \beta \in W^{1,1}\big([0,\infty[;\mathbb{R}\big), \quad \alpha'(t) < 0 \quad \text{and} \quad \beta'(t) > 0. \tag{3.1}$$

As $u(x,t) = u_x(x,t) = 0$ on Σ_t then we have the following identities

$$\begin{aligned} u_t\left(\alpha(t),t\right) &= -\alpha'(t)u_x\left(\alpha(t),t\right) \equiv 0 \quad \text{for all} \quad t \geq 0, \\ u_t\left(\beta(t),t\right) &= -\beta'(t)u_x\left(\beta(t),t\right) \equiv 0 \quad \text{for all} \quad t \geq 0, \\ u_{xt}\left(\alpha(t),t\right) &= -\alpha'(t)u_{xx}\left(\alpha(t),t\right) \quad \text{for all} \quad t \geq 0, \\ u_{xt}\left(\beta(t),t\right) &= -\beta'(t)u_{xx}\left(\beta(t),t\right) \quad \text{for all} \quad t \geq 0. \end{aligned} \tag{3.2}$$

Taking the scalar product on $L^2(\Omega_t)$ of u_t with both sides of (2.1), integrating by parts the second and the third term of left-hand side, observing the identities (3.2) and the Leibnitz's rule[1] yields

$$\frac{d}{dt}E(t) + \frac{1}{2}\beta'(t)u_{xx}^2(\beta(t),t) - \frac{1}{2}\alpha'(t)u_{xx}^2(\alpha(t),t) + \nu\,|u_t(t)|^2 = 0,$$

for all $t \geq 0$ where

$$E(t) = \frac{1}{2}\Big\{\,|u_t(t)|^2 + \zeta_0\,|u_x(t)|^2 + \frac{\zeta_1}{2}\,|u_x(t)|^4 + |u_{xx}(t)|^2\,\Big\}. \tag{3.3}$$

The hypothesis (3.1) yields

$$\frac{1}{2}\beta'(t)u_{xx}^2(\beta(t),t) - \frac{1}{2}\alpha'(t)u_{xx}^2(\alpha(t),t) \geq 0 \quad \text{for all} \quad t \geq 0.$$

Thus, we see that the energy $E(t)$ is not increasing in view of

$$\frac{d}{dt}E(t) \leq -\nu\,|u_t(t)|^2 \quad \text{for all} \quad t \geq 0. \tag{3.4}$$

The asymptotic behavior of the energy $E(t)$ is given by

Theorem 3.1 *The energy $E(t)$ associated with weak solution of the system (2.1)-(2.3), guaranteed by Theorem 2.3, satisfies*

$$E(t) \leq \mathcal{K}\exp(-\omega t) \quad \textit{for all} \quad t \geq 0, \tag{3.5}$$

where ω is a real-positive number that depends on ϵ, $\mathcal{K} = E(0) + \epsilon F(0)$. The function $F(t)$ and the constant ϵ are defined as

$$F(t) = \frac{1}{2}\nu\,|u(t)|^2, \quad \epsilon = \min\left\{\frac{1}{\sqrt{c}},\ \nu\right\}, \tag{3.6}$$

and $c > 0$ is the constant of the immersion of $H_0^2(\Omega_t)$ into $L^2(\Omega_t)$ dependent only of the length of the interval $[\alpha(t),\beta(t)]$ for all $t \geq 0$.

---

[1]
$$\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} f(x,t)dx = \int_{\alpha(t)}^{\beta(t)} \frac{\partial}{\partial t}f(x,t)dx + f(\beta(t),t)\beta'(t) - f(\alpha(t),t)\alpha'(t).$$

Proof. First, we will get (3.5) for the energy $E(t)$ given by the strong solutions of (2.1)-(2.3) guaranteed by Theorem 2.1. We will use the Haraux & Zuazua's method [9], cf. also Zuazua [25]. Consequently, the stability for the energy associated with the weak solution will be established by density arguments.

For each $\epsilon > 0$ we consider the auxiliary function

$$E_\epsilon(t) = E(t) + \frac{\epsilon}{2}\left(u_t(t), u(t)\right) \quad \text{for all} \quad t \geq 0. \tag{3.7}$$

As $H_0^2(\Omega_t)$ is continuously embedded in $L^2(\Omega_t)$ we obtain after the application of usual inequalities that, there exists a constant c such that

$$\frac{\epsilon}{2}\left(u_t(t), u(t)\right) \leq \frac{1}{4}\left|u_t(t)\right| + \frac{\epsilon^2 c}{4}\left|u_{xx}(t)\right|^2. \tag{3.8}$$

Hence and hypothesis $(3.6)_2$ we get

$$E_\epsilon(t) \leq \frac{3}{2}E(t) \quad \text{for all} \quad t \geq 0.$$

On the other hand, we get from (3.7) and (3.8) that

$$\begin{aligned} E_\epsilon(t) \quad &\geq \quad E(t) - \frac{\epsilon}{2}\left|\left(u_t(t), u(t)\right)\right| \\ &\geq \quad \frac{1}{2}E(t) \quad \text{for all} \quad t \geq 0. \end{aligned}$$

Thus, the function E_ϵ satisfies the both inequality

$$\frac{1}{2}E_\epsilon(t) \leq E(t) \leq \frac{3}{2}E_\epsilon(t) \quad \text{for all} \quad t \geq 0. \tag{3.9}$$

Differentiating the function E_ϵ with respect to t, observing the Leibnitz's rule and the boundary conditions (1.2) yields

$$E'_\epsilon(t) = E'(t) + \frac{\epsilon}{2}\left(u_{tt}(t), u(t)\right) + \frac{\epsilon}{2}\left|u_t(t)\right|^2 \quad \text{for all} \quad t \geq 0.$$

Replacing u_{tt} by $\left\{\zeta_0 + \zeta_1 \int_{\Omega_t} |u_x|^2 dx\right\} u_{xx} - u_{xxxx} - \nu u_t$ into the second term of the right-hand side of the last identity preceding, and also using the definition of the function F yields

$$E'_\epsilon(t) + \epsilon F'(t) = E'(t) + \frac{\epsilon}{2}\left|u_t(t)\right|^2 -$$
$$\frac{\epsilon}{2}\left\{\zeta_0\left|u_x(t)\right|^2 + \zeta_1\left|u_x(t)\right|^4 + \left|u_{xx}(t)\right|^2\right\} \quad \text{for all} \quad t \geq 0.$$

Hence, from (3.4) and $(3.6)_2$ we obtain

$$E'_\epsilon(t) + \epsilon F'(t) \leq -\frac{\epsilon}{2}E(t) \quad \text{for all} \quad t \geq 0. \tag{3.10}$$

Using the definitions of the functions $E,\ E_\epsilon$ and F it is easy to see that there exists a real positive constant c_0 dependent only of ϵ such that

$$E_\epsilon(t) + \epsilon F(t) \leq c_0 E(t) \quad \text{for all} \quad t \geq 0. \tag{3.11}$$

Thus, from inequalities (3.10) and (3.11) there exists a suitable real positive constant ω dependent of ϵ such that

$$E'_\epsilon(t) + \epsilon F'(t) + \omega \{E_\epsilon(t) + \epsilon F(t)\} \leq 0 \quad \text{for all} \quad t \geq 0.$$

Therefore,

$$E_\epsilon(t) + \epsilon F(t) \leq \{E_\epsilon(0) + \epsilon F(0)\} \exp(-\omega t) \quad \text{for all} \quad t \geq 0.$$

Hence, and from inequality (3.9) we conclude that the inequality (3.5) holds. Thus, the demonstration of Theorem 3.1 is completed ■

4 Numerical analysis

Our goal in this section is the numeric implementation of approximate solutions v^m of system (2.22) by using both Finite element and Finite difference methods. Besides, some numeric experiments will be presented to analyze the effect of the moving boundary in the vibration of the beam.

4.1 Finite element method

We now present a semi-discrete formulation for problem (2.9)-(2.11) using the Galerkin's method to discretize the spatial variable. Let $\Omega_e = (x_1^e, x_2^e)$ be the partition of the domain $\Omega = (0,1)$ such that

$$\Omega = \text{int}\left(\bigcup_{e=1}^{m} \bar{\Omega}_e\right) \quad \text{and} \quad \Omega_e \cap \Omega_s = \emptyset, \quad \text{if} \quad e \neq s$$

For a given integer $l \geq 1$, we introduce the finite element space

$$\mathcal{N}_h^l(\Omega) = \left\{ q_h \in C^2(\Omega);\ q_{h|\Omega_e} \ \in \ P_l(\Omega_e) \right\},$$

where $P_l(K)$ is the set of polynomials on Ω_e of degree less than or equal to l, i.e., $\mathcal{N}_h^l$ is the space of piecewise continuous polynomial functions of degree l. More specifically, we are using as base function the B-splines that are cubic splines defined in the following way

$$\theta_i(y) = \begin{cases} \dfrac{1}{4h^3}(y-y_{i-2})^3 \quad \text{with} \quad y_{i-2} \le y \le y_{i-1}, \\ \dfrac{1}{4} + \dfrac{3}{4h}(y-y_{i-1}) + \dfrac{3}{4h^2}(y-y_{i-1})^2 - \\ \dfrac{3}{4h^3}(y-y_{i-1})^3 \quad \text{with} \quad y_{i-1} \le y \le y_i, \\ \dfrac{1}{4} + \dfrac{3}{4h}(y_{i+1}-y) + \dfrac{3}{4h^2}(y_{i+1}-y)^2 - \\ \dfrac{3}{4h^3}(y_{i+1}-y)^3 \quad \text{with} \quad y_i \le y \le y_{i+1}, \\ \dfrac{1}{4h^3}(y_{i+2}-y)^3 \quad \text{with} \quad y_{i+1} \le y \le y_{i+2}, \end{cases} \tag{4.1}$$

where the node points are equally spaced with spacing $h = y_{i+1} - y_i$, $y_1 = 0$ and $y_m = 1$. We would like to use the basis functions $w_1, \cdots, w_m$ that define (2.20) instead of the B-splines $\theta_1, \cdots, \theta_m$. However, although $\theta_3, \cdots, \theta_{m-2}$ satisfy the zero boundary conditions, θ_1, θ_2, θ_{m-1} and θ_m do not. Therefore we define the w_i for

$$\begin{aligned} &w_i(y) = \theta_i(y) \quad i = 3, \cdots, m-2, \\ &w_1(y) = 0, \\ &w_2(y) = \theta_0(y) - \tfrac{1}{2}\theta_1(y) + \theta_2(y), \\ &w_{m-1}(y) = \theta_{m-2}(y) - \tfrac{1}{2}\theta_{m-1}(y) + \theta_m(y), \\ &w_m(y) = 0. \end{aligned} \tag{4.2}$$

From (4.2) we can verify that $w_1(0) = w_2(0) = w_{m-1}(1) = w_m(1) = 0$ and $w_i(y)$ is a cubic spline that satisfies the boundary condition (2.11).

Substituting (4.1) into (2.22) and denoting ϕ by $w_j(y)$ and $\frac{\partial^k}{\partial y^k}$ by $\partial^k{}_y$ for $k = 1, 2,$ we get

$$\begin{gathered} g''_{im}(t)\left(\phi_i(y), \phi_j(y)\right) + g_{im}(t)\left(a_1(y,t)\partial_y\phi_i(y), \phi_j(y)\right) + \\ g'_{im}(t)\left(a_2(y,t)\partial_y\phi_i(y), \phi_j(y)\right) - g_{im}(t)\left(\partial_y a_3(y,t)\partial_y\phi_i(y), \phi_j(y)\right) + \\ b(t)g_{im}(t)\left(\partial_y\phi_i(y), \partial_y\phi_j(y)\right) + b_2(t)g_{im}(t)\left(\partial_y^2\phi_i(y), \partial_y^2\phi_j(y)\right) + \\ \nu g'_{im}(t)\left(\phi_i(y), \phi_j(y)\right), \end{gathered} \tag{4.3}$$

where

$$b(t) = b_1(t)\left(\partial_y v(y), \partial_y v(y)\right).$$

Making some modifications in (4.3) we can write

$$\begin{aligned} & g''_{im}(t)\,[L_{i,j}] + g'_{im}(t)\,[a_2(y,t)P_{i,j} + \nu L_{i,j}] + \\ & g_{im}(t)\left[\left(a_1(y,t) + \frac{\gamma'(t)}{\gamma^2(t)}a_2(y,t)\right)P_{i,j}\right] - \\ & g_{im}(t)\Big[\big(a_3(y,t) + b(t)\big)M_{i,j} + b_2(t)N_{i,j}\Big] = 0, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} [L_{i,j}] &= (\phi_i(y), \phi_j(y)), \quad [M_{i,j}] = (\partial_y\phi_i(y), \partial_y\phi_j(y)), \\ [N_{i,j}] &= \big(\partial_y^2\phi_i(y), \partial_y^2\phi_j(y)\big), \quad [P_{i,j}] = (\partial_y\phi_i(y), \phi_j(y)). \end{aligned} \tag{4.5}$$

Note that in deriving equation (4.4) the coefficients are approximated by piecewise constants in each element e defined in Ω_e sufficiently small such that y is taken constant on each element Ω_e, once the coefficients are continuous on $(0,1)$.

The matrix (4.5) are explicitly obtained thanks to (4.1). Hence we obtain the following system of second order ordinary differential equations

$$Lg''_{im}(t) + Cg'_{im}(t) + Kg_{im}(t) = 0, \tag{4.6}$$

where

$$\begin{aligned} & L = [L_{i,j}], \quad C = [a_2(y,t)P_{i,j} + \nu L_{i,j}] \quad \text{and} \\ & K = \left[\left(a_1(y,t) + \frac{\gamma'(t)}{\gamma^2(t)}a_2(y,t)\right)P_{i,j} - \big(a_3(y,t) + b(t)\big)M_{i,j} + b_2(t)N_{i,j}\right]. \end{aligned} \tag{4.7}$$

4.2 Finite difference method

As the matrix (4.7) depends on the variables y and t is not always possible to obtain the exact solution. Being like this, we will apply the Newmark's method, see for instance, Hughes [10], pp 493 and also Newmark [21], to obtain the approximate solutions of the system (4.6). Thus, for each instant of discrete time $t_n = n\Delta t$, we consider

$$\begin{aligned} & La_{n+1} + Cv_{n+1} + Kd_{n+1} = 0, \\ & d_{n+1} = d_n + \Delta t v_n + \frac{\Delta t^2}{2}a_n, \\ & v_{n+1} = v_n + \frac{\Delta t}{2}(a_n + a_{n+1}), \end{aligned} \tag{4.8}$$

where a_n, v_n and d_n represent $d''(t_n)$, $d'(t_n)$ and $d(t_n)$ respectively. For each representation we can show that the order of the error of discretization in the Taylor's expansion is $o(\Delta t^3)$.

4.3 Implementation

From initial conditions, d_0 and $d_0' = v_0$ are known. Then taking $t = 0$ into $(4.8)_1$ yields

$$La_0 = -Cv_0 - Kd_0.$$

and hence we can get a_0.

For $n = 0, 1 \cdots$ we have

$$\left(L + \frac{\Delta t}{2} C\right) a_{n+1} = -C\tilde{v}_{n+1} - K\tilde{d}_{n+1}, \tag{4.9}$$

where the predictors are given by

$$\begin{aligned} \tilde{d}_{n+1} &= d_n + \Delta t v_n + \frac{\Delta t^2}{2} a_n, \\ \tilde{v}_{n+1} &= v_n + \tfrac{\Delta t}{2} a_n. \end{aligned}$$

To determine the values of d_{n+1} and v_{n+1} we do

$$\begin{aligned} d_{n+1} &= \tilde{d}_{n+1}, \\ v_{n+1} &= \tilde{v}_{n+1} + \frac{\Delta t^2}{2} a_{n+1}. \end{aligned}$$

The values of d_n are calculated in each instant of time $t_n = n\Delta t$ with $0 \le t_n \le T$. Substituting the values of d_n in the equation (4.9) we obtain the approximate solutions of the problem (2.22) on the subspace V_m. As the application $\mathcal{T}$ is an isomorphism then the solutions of the problem (2.22) are also solutions of the problem (2.1)-(2.3).

4.4 Numerical Simulation

Two numeric examples will be shown in this subsection to illustrate the characteristics of the beam equation with fixed and moving boundaries. In these examples will be used 20 finite element, e.g., $\Delta y = \frac{1}{20}$, and in the discrete time $\Delta t = 0.0005$ making 10000 iterations.

Example 1 In this example we consider the fixed domain $\alpha(t) = 0$ and $\beta(t) = 1$. The initial position and initial velocity are given by

$$v(y, 0) = \frac{1}{\pi^2} y^2 (y-1)^2,$$

$$v_t(y, 0) = 0.$$

The figures 1 and 2 show the effect of the variation of ζ_1 in the vibrations of the beam. We consider $\zeta_1 = 1$ and $\zeta_1 = 10$ respectively, and in both $\zeta_0 = 1$ and $\nu = 0$.

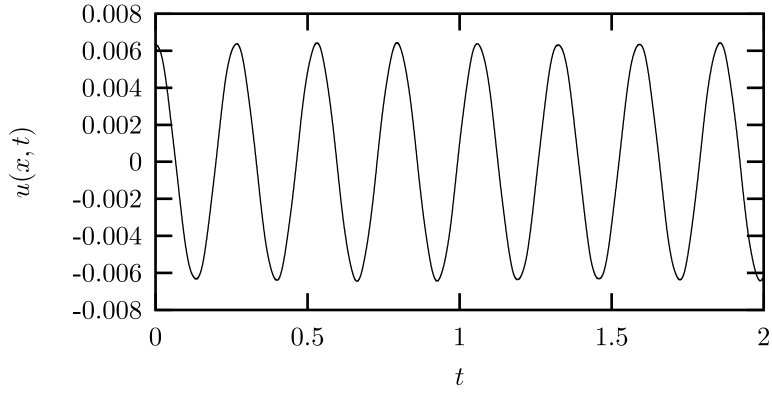

Figure 1

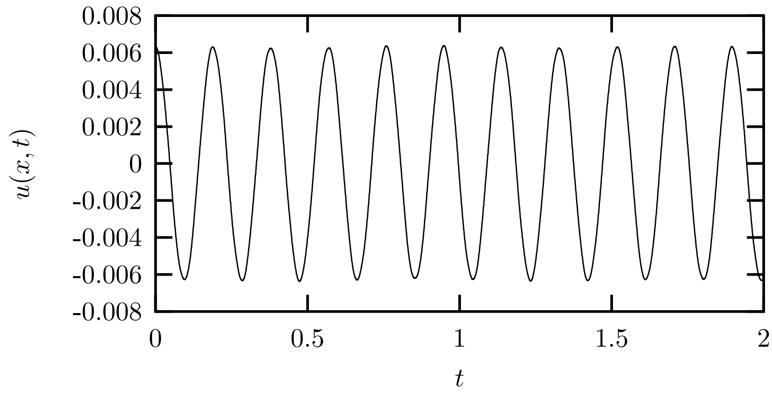

Figure 2

Example 2 In this example the boundary is moving, namely $\alpha(t) = -\frac{t}{t+1}$ and $\beta(t) = \frac{2t+1}{t+1}$. Both initial position and initial velocity are the same of the example 1. We take also $\zeta_0 = 1$ and $\zeta_1 = 5$. The figure 3 shows the position of the beam in some instant of time. This way we can see the increasing of the length of the interval $\gamma(t)$.

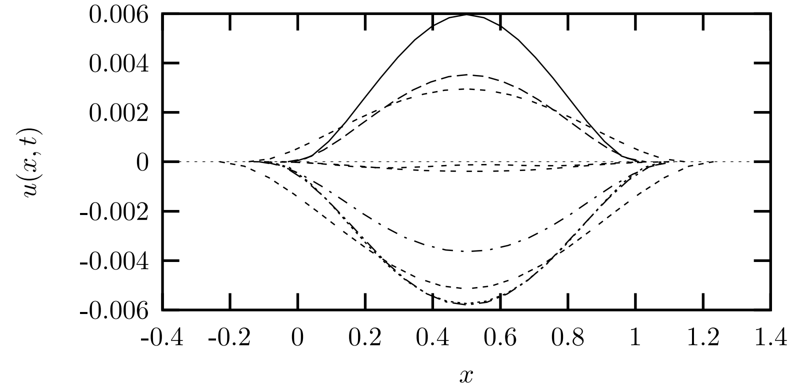

Figure 3

The figures 4, 5 and 6 show the influence of the internal damping constant ν in the medium nodal of the beam, i.e., it shows the approximate solution $v^m(0.5,t)$ when we take $\nu = 0$, $\nu = 0.5$ and $\nu = 1$. We can verify that the oscillation of the beam decay faster when ν increases.

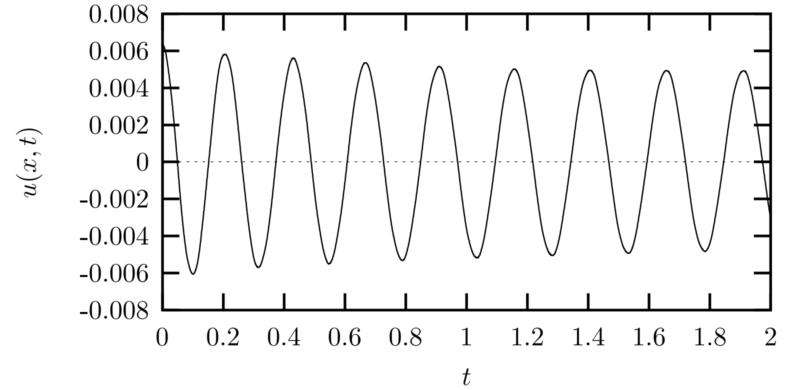

Figure 4: $\nu = 0$

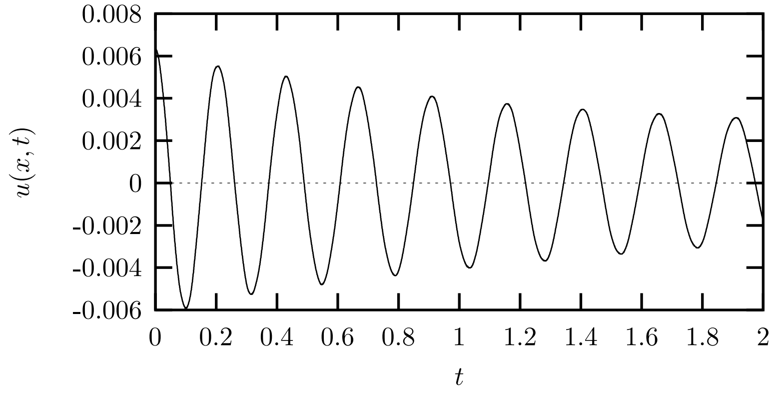

Figure 5: $\nu = 0.5$

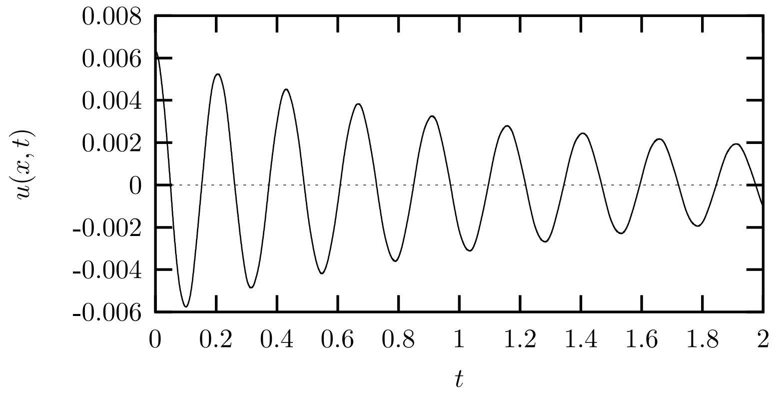

Figure 6: $\nu = 1.0$

Finally we present the figure 7 that shows the oscillation of the beam $u(x,t)$ when the boundary is defined by $\alpha(t) = -\frac{t}{t+1}$ and $\beta(t) = \frac{2t+1}{t+1}$.

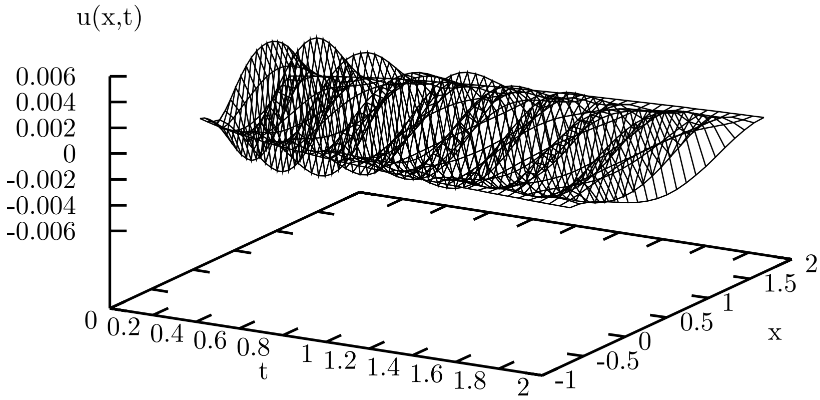

Figure 7

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