Error Estimates of Vibrations of Elastic String with Moving Ends

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Abstract

In this work, a mathematical model for small vibrations of an elastic string with moving boundaries is discussed. Global existence and uniqueness of solutions have been established in [12]. The objective of this paper is to obtain the error estimates of solutions in Sobolev spaces for the semi-discrete problem, with discretization of space variable and continuous time. The analysis is based on Galerkin finite element method.

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1 Introduction

The Kirchhoff equation, modeling the vibration of an elastic string, is given by [6]

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{m}\left\{\tau_0 + \frac{\kappa}{2L_0}\int_\alpha^\beta \left(\frac{\partial u}{\partial x}\right)^2 dx\right\}\frac{\partial^2 u}{\partial x^2} = 0, \tag{1}$$

where $u = u(x,t)$ is the transverse displacement of the string, m is the mass per unit length, τ_0 is the tension, $L_0 = \beta - \alpha$ is the length of the string at rest and κ is the Young's modulus. The extension of Kirchhoff's model to moving boundaries was developed by Medeiros *et al.* [9, 10], where the existence and local uniqueness were analyzed. The operator for this problem is defined by

$$\widehat{L}u = \frac{\partial^2 u}{\partial t^2} - \Big(\widetilde{a}(t) + \widetilde{b}\int_{\alpha(t)}^{\beta(t)} \Big(\frac{\partial u}{\partial x}\Big)^2 dx\Big)\frac{\partial^2 u}{\partial x^2}, \tag{2}$$

$$\alpha(t) < x < \beta(t), \qquad t > 0,$$

where $\alpha(t)$ and $\beta(t)$ are the positions of the left and the right ends at the instant t, respectively. The horizontal length of the string is defined by $\gamma(t) = \beta(t) - \alpha(t) > 0$.

The two coefficients $\widetilde{a}(t)$ and $\widetilde{b}$ depend on the elastic properties and are given by[1]

$$\widetilde{a}(t) = \frac{\tau_0}{m} + \frac{\kappa}{m}\frac{\gamma(t) - L_0}{L_0}, \qquad \widetilde{b} = \frac{\kappa}{2mL_0}. \tag{3}$$

The existence and uniqueness of solutions for this nonlinear case have been studied in [9] and numerical simulations with finite difference method have been considered in [8].

In (1), the nonlinear term comes from the change of length of the string due to the transverse displacement in the vibration. If such a change of length is insignificant, the nonlinear term in (2) can be neglected and consequently, the nonlinear operator $\widehat{L}(\cdot)$ reduces to a linear one,

$$\widetilde{L}u(x,t) = \frac{\partial^2 u}{\partial t^2} - \widetilde{a}(t)\frac{\partial^2 u}{\partial x^2}. \tag{4}$$

However, the tension is not constant and, therefore, $\widetilde{a}(t)$ is not constant in general because of the moving ends.

We emphasize that, although the term multiplied by $\widetilde{b}$ is neglected, this is different from assuming $\widetilde{b} = 0$ (hence $\kappa/m = 0$), because otherwise, we would require $\widetilde{a}(t)$ to be a constant from (3). In particular, if $\widetilde{a}(t)$ is constant in (4) with moving boundary at one of the ends, i.e,

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{for} \qquad 0 \le x \le vt, \tag{5}$$

the proofs of uniqueness and existence of the solution and asymptotic behavior analysis have been carried out in [1], [3] and [5].

We shall formulate the variational problem of (4) by the Galerkin method:

$$\textbf{(I)} \quad \begin{cases} \widetilde{L}u(x,t) = f(x,t) & \forall\,(x,t) \in \widehat{Q}, \\ u(x,t) = 0 & \forall\,(x,t) \in \widehat{\Sigma}, \\ u(x,0) = u_0(x), & \dfrac{\partial u}{\partial t}(x,0) = u_1(x); \quad \alpha(0) < x < \beta(0), \end{cases} \tag{6}$$

where $\widehat{Q} = \big\{(x,t) \in I\!R^2;\ \alpha(t) < x < \beta(t),\ t > 0\big\}$ is the non-cylindrical domain with boundary $\widehat{\Sigma} = \bigcup\limits_{0<t<T} \{(\alpha(t), \beta(t))\} \times \{t\}$.

In [12], global existence and local uniqueness of the solution of (6) have been proved. A summary of results in [12] will be given in Section 2.

In this paper, based on the same hypotheses used in [12] (see H1 and H2 below), we shall establish the error estimates of the solution in Sobolev spaces for the semi-discrete problem with the finite element method.

[1]This corresponds to the model proposed in [9] where $\tilde{b} = b(t) = k/2m\gamma(t)$ was derived (for this difference, see the comment in Footnote 1 of [8]).

2 The Continuous Problem

In order to solve problem (I) we shall consider an equivalent problem defined in a fixed domain by the following change of variables,

$$\begin{aligned} \mathcal{T} \,:\, Q_t &\to Q = (0,1)\times(0,T) \\ (x,t) &\mapsto (y,t) = \Big(\frac{x-\alpha(t)}{\gamma(t)}\;,\; t\Big), \end{aligned} \tag{7}$$

which is C^2. The inverse $\mathcal{T}^{-1}$ is also C^2. Using the application (7) and notations

$$f(x,t) = f\left(\mathcal{T}^{-1}(y,t)\right) = g(y,t),$$

$$u(x,t) = u\left(\mathcal{T}^{-1}(y,t)\right) = v(y,t),$$

we find from (4) the operator

$$Lv(y,t) = \frac{\partial^2 v}{\partial t^2} + a(y,t)\frac{\partial^2 v}{\partial y^2} + b(y,t)\frac{\partial^2 v}{\partial t \partial y} + c(y,t)\frac{\partial v}{\partial y}, \tag{8}$$

with

$$a(y,t) = \frac{1}{4}\, b(y,t)^2 - \frac{1}{\gamma(t)^2}\, \widetilde{a}(t), \tag{9}$$

$$b(y,t) = -2\, \frac{\alpha'(t)+\gamma'(t)y}{\gamma(t)}, \tag{10}$$

$$c(y,t) = -\frac{1}{\gamma(t)}\Big(\alpha''(t) + \gamma''(t)y + \gamma'(t)\, b(y,t)\Big), \tag{11}$$

where $\widetilde{a}(t)$ is defined in (3). Thus, for the rectangular domain Q we have the equivalent problem:

$$\textbf{(II)} \quad \left\{ \begin{array}{ll} Lv(y,t) = g(y,t) & \forall\,(y,t) \in (0,1)\times(0,T), \\ v(0,t) = v(1,t) = 0, & 0 < t < T, \\ v(y,0) = v_0(y), & \dfrac{\partial v}{\partial t}(y,0) = v_1(y), \quad 0 \le y \le 1. \end{array} \right. \tag{12}$$

Owing to the change of variables $\mathcal{T}$, $u(x,t)$ is a solution of the problem (I) if and only if $v(y,t)$ is a solution of the problem (II).

Let $\|\,.\,\|$ represents the norm in $H_0^1(0,1)$ and $(\;,\;)$, $|\,.\,|$ be the scalar product and the norm in $L^2(0,1)$ respectively. Consider the hypotheses:

H1: $\alpha, \beta \in C^3\left([0,T]; I\!R\right)$.

H2: $a(y,t) < 0, \quad \forall\,(y,t) \in (0,1)\times[0,T]$.

In [12], we have shown the following result for problem (II) and problem (I):

Theorem 1 (Strong Solution) *Under the hypotheses* (H1) *and* (H2) *and given the initial data*

$$v_0 \in H_0^1(0,1) \cap H^2(0,1), \quad v_1 \in H_0^1(0,1), \quad g, \frac{\partial g}{\partial t} \in L^2((0,T); L^2(0,1)),$$

for all $T > 0$*, there exists a unique solution* $v : Q \to I\!R$*, satisfying the following conditions:*

1. $v \in L^\infty((0,T); H_0^1(0,1) \cap H^2(0,1))$,
2. $\dfrac{\partial v}{\partial t} \in L^\infty((0,T); H_0^1(0,1))$,
3. $\dfrac{\partial^2 v}{\partial t^2} \in L^\infty((0,T); L^2(0,1))$,
4. $\dfrac{\partial^2 v}{\partial t^2} + a(y,t)\dfrac{\partial^2 v}{\partial y^2} + b(y,t)\dfrac{\partial^2 v}{\partial t \partial y} + c(y,t)\dfrac{\partial v}{\partial y} = g(y,t), \quad a.e. \ in \ Q,$
5. $v(0) = v_0, \quad \dfrac{\partial v}{\partial t}(0) = v_1.$

Moreover, we also have the following theorem for Problem (I):

Theorem 2 (Strong Solution) *Let* $\Omega_t = (\alpha(t), \beta(t))$*,* $\Omega_0 = (\alpha(0), \beta(0))$ *and the initial data*

$$u_0 \in H_0^1(\Omega_0) \cap H^2(\Omega_0), \quad u_1 \in H_0^1(\Omega_0), \quad f, \frac{\partial f}{\partial t} \in L^2((0,T); L^2(\Omega_t)),$$

for all $T > 0$*. Under the hypotheses* (H1) *and* (H2)*, there exists a unique solution of problem* (I)*,* $u : \widehat{Q} \to I\!R$*, satisfying the following conditions:*

1. $u \in L^\infty((0,T); H_0^1(\Omega_t) \cap H^2(\Omega_t))$,
2. $\dfrac{\partial u}{\partial t} \in L^\infty((0,T); H_0^1(\Omega_t))$,
3. $\dfrac{\partial^2 u}{\partial t^2} \in L^2((0,T); L^2(\Omega_t))$,
4. $\dfrac{\partial^2 u}{\partial t^2} - \widetilde{a}(t)\dfrac{\partial^2 u}{\partial x^2} = f(x,t), \quad a.e. \ in \ \widehat{Q},$
5. $u(0) = u_0, \quad \dfrac{\partial u}{\partial t}(0) = u_1, \quad in \ \Omega_0.$

2.1 Variational Formulation

Let $a(t,v,w)$ and $\hat{a}(t,v,w)$ be bilinear forms in $H_0^1(0,1)$ defined by

$$a(t,v,w) = \int_0^1 a(y,t)\frac{\partial v}{\partial y}\frac{\partial w}{\partial y}\,dy, \tag{13}$$

$$\hat{a}(t,v,w) = \int_0^1 \frac{\partial a}{\partial y}(y,t)\frac{\partial v}{\partial y}w\,dy. \tag{14}$$

Then, the variational formulation of problem (II) can be expressed as

$$\begin{cases} \Big(\dfrac{\partial^2 v}{\partial t^2}, w\Big) + \Big(a(y,t)\dfrac{\partial^2 v}{\partial y^2}, w\Big) + \Big(b(y,t)\dfrac{\partial^2 v}{\partial t\partial y}, w\Big) \\ \qquad + \Big(c(y,t)\dfrac{\partial v}{\partial y}, w\Big) = \Big(g, w\Big), \qquad \forall\, w \in H_0^1(0,1), \\ u(0) = u_0(x), \qquad u'(x,0) = u_1(x), \qquad \forall\, x \in (0,1). \end{cases} \tag{15}$$

The second term on the left hand side of (15) can be rewritten as

$$\int_0^1 a(y,t)\frac{\partial^2 v}{\partial y^2}\;w\;dy = -\int_0^1 \frac{\partial v}{\partial y}\frac{\partial}{\partial y}(a(y,t)w)\;dy = -\int_0^1 \frac{\partial a}{\partial y}(y,t)\frac{\partial v}{\partial y}w\;dy$$
$$-\int_0^1 a(y,t)\frac{\partial v}{\partial y}\frac{\partial w}{\partial y}\;dy = -\hat{a}(t,v,w) - a(t,v,w). \tag{16}$$

To obtain (16) we have used the definitions (13), (14) and that $w \in H_0^1(0,1)$. Thus, combining (15) with (16), we have a weak formulation for problem (II) consisting in finding $v(t) \in H_0^1(0,1)$, for each $0 < t < T$, such that (we suppress the dependence of the coefficients on y)

$$\Big(\frac{\partial^2 v}{\partial t^2}, w\Big) - \hat{a}(t,v,w) - a(t,v,w) + \Big(b(t)\frac{\partial^2 v}{\partial t\partial y}, w\Big) + \Big(c(t)\frac{\partial v}{\partial y}, w\Big) = \Big(g, w\Big), \tag{17}$$

for all $w \in H_0^1(0,1)$. In the next section we will consider the continuous time approximation of $v(t)$ to the solution of (17).

3 The Semi-Discrete Problem

We now present a semi-discrete formulation for problem (II) using the Galerkin finite element method to discretize the space variable. Let $\{\mathcal{T}_h\}$ be a family of polygonalization $\mathcal{T}_h = \{K\}$ of Ω, satisfying the minimum angle condition (see for instance [2]) and indexed by the parameter h representing the maximum diameter of elements $K \in \mathcal{T}_h$. For a given integer $l \geq 1$, we introduce the finite element space

$$\mathcal{N}_h^l = \left\{q_h \in C^0(0,1);\; q_h|_K \;\in\; P_l(K),\; \forall\; K \in\; \mathcal{T}_h\right\},$$

where $P_l(K)$ is the set of polynomials on K of degree less than or equal to l, i.e., $\mathcal{N}_h^l$ is the space of piecewise continuous polynomial functions of degree l. By standard interpolation theory (see for instance [2]), it follows that given a function $w : (0,T) \longrightarrow H^{l+1}(0,1)$ there exists an interpolator $\hat{w}_h : (0,T) \longrightarrow \mathcal{N}_h^l$ such that

$$\|w(t) - \hat{w}_h(t)\|_m \leq C_1 h^{l+1-m} \|w(t)\|_{l+1}, \tag{18}$$

where $\|\cdot\|_m$ denotes the usual semi-norm in the Hilbert space $H^m(0,1)$ and $m \leq l$. In the following analysis, we shall use the above result for $m = 1$ and $l = 1$ and write $\mathcal{N}_h^l$ simply as $\mathcal{N}_h$.

The Galerkin finite element semi-discrete approximation of problem (17) in $\mathcal{N}_h$ reads

$$\text{Find} \qquad v_h : (0,T) \longrightarrow \mathcal{N}_h \qquad \text{such that}$$

$$\begin{aligned} &\Big(\frac{\partial^2 v_h}{\partial t^2}(t), w_h\Big) - \hat{a}\Big(t, v_h(t), w_h\Big) - a\Big(t, v_h(t), w_h\Big) \\ &+\Big(b(t)\ \frac{\partial^2 v_h(t)}{\partial t \partial y}, w_h\Big) + \Big(c(t)\ \frac{\partial v_h(t)}{\partial y}, w_h\Big) = \Big(g, w_h\Big), \quad \forall w_h \in \mathcal{N}_h. \end{aligned} \tag{19}$$

Let

$$e(t) = v(t) - v_h(t), \quad t \in (0,T), \tag{20}$$

be the error associated with the semi-discrete problem (19). We will establish the error estimates in the $H^1(0,1)$-norm due to the space-discretization.

3.1 Error Estimates

In this section we investigate the error estimates in Sobolev spaces for the problem (II). For this we need the following additional hypotheses for the interpolators of the initial data, i.e,

H3: $\qquad \|v_h(0) - \hat{v}_0\| \leq \bar{c}\, h; \qquad |v_h'(0) - \hat{v}_0'| \leq \widehat{c}\, h,$

where $\hat{v}_0$ and $\hat{v}_0'$ are the interpolators of the initial conditions v_0 and v_0' and $\bar{c}$ and $\widehat{c}$ are positive constants independent of h. For convenience, we have also used the prime ($'$) to denote the derivative with respect to time t.

Theorem 3 *Under the hypotheses H1, H2 and H3, and given the initial data*

$$v_0 \in H_0^1(0,1) \cap H^2(0,1), \quad v_1 \in H_0^1(0,1), \quad g, g' \in L^2((0,T); L^2(0,1)),$$

there exists $T > 0$ and a unique solution of Problem (II) *$v : Q \to I\!R$, satisfying the following conditions*

$$|v' - v_h'|_{L^\infty((0,T);L^2(0,1))} + \|v - v_h\|_{L^\infty((0,T);H_0^1(0,1))} \leq Ch. \tag{21}$$

where C is a positive constant independent of h.

Proof. We split equation (20) into two parts

$$e(t) = \rho(t) + \theta(t), \tag{22}$$

where $\rho(t) = v(t) - \hat{v}_h(t)$ and $\theta(t) = \hat{v}_h(t) - v_h(t)$. Following the ideas in the elliptic approximation theory, let $\hat{v}_h(t) \in \mathcal{N}_h$ be the elliptic projection of $v(t)$ into $\mathcal{N}_h$, for each fixed $t \in (0, T)$, defined by

$$a\big(t; \hat{v}_h(t), w_h\big) = a\big(t; v(t), w_h\big), \quad \forall w_h \in \mathcal{N}_h. \tag{23}$$

Then, $\hat{v}_h(t)$ can be viewed as an interpolator of $v(t)$. From equations (18) and (23) it is possible to make the error estimates for $\rho(t)$

$$|\rho(t)| + h\|\rho(t)\| \leq C_1 h^2 \|v(t)\|_2. \tag{24}$$

Analogous results can be obtained for $\rho'(t)$ and $\rho''(t)$ if we assume that $v(t)$ and $\hat{v}_h(t)$ are sufficiently regular.

Thus, it remains to estimate the variable in $\mathcal{N}_h$, i.e., $\theta(t)$. First, from the assumptions on the coefficients $a(y,t)$ and $b(y,t)$, we arrive at

$$\Big|\frac{\partial a}{\partial y}(y,t)\Big| = \frac{2}{\gamma^2}|\gamma'|\,|\alpha' + \gamma' y| \leq C_2, \tag{25}$$

$$\Big|\frac{\partial a}{\partial t}(y,t)\Big| = \Big|\frac{-2(\alpha' + \gamma' y)}{\gamma^3}\Big|\,|\gamma'(\alpha' + \gamma' y) - \gamma(\alpha'' + \gamma'' y)| \leq C_3, \tag{26}$$

and, similarly, we find bounds to $|\dfrac{\partial b}{\partial y}(y,t)|$ and $|c(y,t)|$.

Now, since $\mathcal{N}_h \subset H_0^1(0,1)$ for all $w_h \in \mathcal{N}_h$, we subtract the continuous equation (17) from the semi-discrete one (19), obtaining

$$\begin{aligned}
&\Big(\frac{\partial^2 (v - v_h)}{\partial t^2}, w_h\Big) - \hat{a}\big(t, v - v_h, w_h\big) - a\big(t, v - v_h, w_h\big) \\
&\quad + \Big(b(t)\,\frac{\partial^2 (v - v_h)}{\partial t \partial y}, w_h\Big) + \Big(c(t)\,\frac{\partial (v - v_h)}{\partial y}, w_h\Big) = 0.
\end{aligned} \tag{27}$$

Then, summing and subtracting the interpolator $\hat{v}_h(t)$ from each term in (27) and using (22) and (23), we find

$$\begin{aligned}
&\Big(\frac{\partial^2 \rho}{\partial t^2}, w_h\Big) + \Big(\frac{\partial^2 \theta}{\partial t^2}, w_h\Big) - \hat{a}\big(t, \rho, w_h\big) - a\big(t, \rho, w_h\big) - \hat{a}\big(t, \theta, w_h\big) - a\big(t, \theta, w_h\big) \\
&+ \Big(b(t)\frac{\partial^2 \rho}{\partial t \partial y}, w_h\Big) + \Big(b(t)\frac{\partial^2 \theta}{\partial t \partial y}, w_h\Big) + \Big(c(t)\frac{\partial \rho}{\partial y}, w_h\Big) + \Big(c(t)\frac{\partial \theta}{\partial y}, w_h\Big) = 0.
\end{aligned} \tag{28}$$

In the following we will estimate the approximation error in $\mathcal{N}_h$, i.e., we will find bounds to $\theta(t)$. By taking $w_h = \theta'(t)$ in equation (28) we get

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}|\theta'(t)|^2 - a(t, \theta, \theta') + \Big(b(t)\frac{\partial \theta'}{\partial y}, \theta'\Big) + \Big(c(t)\frac{\partial \theta}{\partial y}, \theta'\Big) = -(\rho'', \theta') \\
&\quad + \hat{a}(t, \theta, \theta') + \hat{a}(t, \rho, \theta') - \Big(b(t)\frac{\partial \rho'}{\partial y}, \theta'\Big) - \Big(c(t)\frac{\partial \rho}{\partial y}, \theta'\Big).
\end{aligned} \tag{29}$$

The second term on the left hand side of (29) can be rewritten as

$$-a\big(t,\theta,\theta'\big) = -\int_0^1 a(y,t)\frac{\partial\theta}{\partial y}\frac{\partial\theta'}{\partial y}\,dy = -\frac{1}{2}\int_0^1 a(y,t)\frac{\partial}{\partial t}\Big(\frac{\partial\theta}{\partial y}\Big)^2\,dy$$
$$= -\frac{1}{2}\frac{\partial}{\partial t}\int_0^1 a(y,t)\Big(\frac{\partial\theta}{\partial y}\Big)^2\,dy + \frac{1}{2}\int_0^1\Big(\frac{\partial a}{\partial t}(y,t)\Big)\Big(\frac{\partial\theta}{\partial y}\Big)^2\,dy. \tag{30}$$

From the generalized theorem of integrals, there exists $\xi\in[0,1]$ such that

$$-\frac{1}{2}\frac{\partial}{\partial t}\int_0^1 a(y,t)\Big(\frac{\partial\theta}{\partial y}\Big)^2\,dy = -\frac{1}{2}\frac{\partial}{\partial t}\Big(a(\xi,t)\|\theta(t)\|^2\Big). \tag{31}$$

Then, substituting (31) from (30), we have

$$-a\big(t,\theta,\theta'\big) = -\frac{1}{2}\frac{\partial}{\partial t}\Big(a(\xi,t)\|\theta\|^2\Big) + \frac{1}{2}\int_0^1\Big(\frac{\partial a}{\partial t}(y,t)\Big)\Big(\frac{\partial\theta}{\partial y}\Big)^2\,dy. \tag{32}$$

Moreover, taking into account the third term in the left hand side of (29), we obtain

$$\big(b(t)\frac{\partial\theta'}{\partial y},\theta'\big) = \int_0^1 b(y,t)\frac{\partial\theta'}{\partial y}\theta'\,dy = \frac{1}{2}\int_0^1 b(y,t)\frac{\partial}{\partial y}(\theta'(t))^2dy$$
$$= \frac{1}{2}(\theta')^2 b(y,t)\Big|_0^1 - \frac{1}{2}\int_0^1\frac{\partial b}{\partial y}(y,t)(\theta')^2dy = -\frac{1}{2}\int_0^1\frac{\partial b}{\partial y}(y,t)(\theta')^2dy. \tag{33}$$

Now, combining (32) and (33) with (29), we arrive at

$$\frac{1}{2}\frac{\partial}{\partial t}|\theta'(t)|^2 - \frac{1}{2}\frac{\partial}{\partial t}\Big(a(\xi,t)\|\theta\|^2\Big) = -\frac{1}{2}\int_0^1\Big(\frac{\partial a}{\partial t}(y,t)\Big)\Big(\frac{\partial\theta(t)}{\partial y}\Big)^2\,dy$$
$$+\quad \frac{1}{2}\int_0^1\frac{\partial b}{\partial y}(y,t)(\theta'(t))^2dy - \int_0^1 c(y,t)\frac{\partial\theta}{\partial y}\theta'(t)dy - (\rho''(t),\theta'(t)) +$$
$$\hat{a}(t,\theta,\theta') + \hat{a}(t,\rho,\theta') - \big(b(y,t)\frac{\partial\rho'}{\partial y},\theta'\big) - \big(c(y,t)\frac{\partial\rho}{\partial y},\theta'\big). \tag{34}$$

Integrating (34) from 0 to t, with $t\le T$ we obtain

$$\frac{1}{2}|\theta'(t)|^2 - \frac{1}{2}a(\xi,t)\|\theta(t)\|^2 = \frac{1}{2}|\theta'(0)|^2 + \frac{1}{2}\,a(\xi,0)\|\theta(0)\|^2$$
$$+\int_0^t\Big\{-\frac{1}{2}\int_0^1\Big(\frac{\partial a}{\partial t}(y,t)\Big)\Big(\frac{\partial\theta}{\partial y}\Big)^2\,dy + \frac{1}{2}\int_0^1\frac{\partial b}{\partial y}(y,t)(\theta')^2dy$$
$$-\int_0^1 c(y,t)\frac{\partial\theta}{\partial y}\,\theta'(t)dy - (\rho'',\theta') + \hat{a}(t,\theta,\theta')$$
$$+\hat{a}(t,\rho,\theta') - \big(b(y,t)\frac{\partial\rho'}{\partial y},\theta'\big) - \big(c(y,t)\frac{\partial\rho}{\partial y},\theta'\big)\Big\}\,dt. \tag{35}$$

Next, we will analyze the right hand side of (35). Using the Schwartz inequality and the bounds exhibited by (25) and (26) we conclude that

$$\int_0^1 \Big(\frac{\partial a}{\partial t}(y,t)\Big)(\frac{\partial \theta}{\partial y})^2\, dy \le c_1 \|\theta(t)\|^2, \tag{36}$$

$$\int_0^1 \frac{\partial b}{\partial y}(y,t)(\theta')^2 dy \le c_2 |\theta'(t)|^2. \tag{37}$$

In the following the inequality $|(\psi,\theta)| \le \lambda\|\psi\|^2 + \frac{1}{\lambda}\|\theta\|^2$, $\lambda > 0$, is extensively used to write

$$\hat{a}(t,\theta,\theta') = \int_0^1 \frac{\partial a}{\partial y}(y,t)\frac{\partial \theta}{\partial y}\,\theta'\, dy \le c_3\Big(\|\theta(t)\|^2 + |\theta'(t)|^2\Big), \tag{38}$$

$$\int_0^1 c(y,t)\frac{\partial \theta}{\partial y}\,\theta'(t) dy \le c_4\Big(\|\theta(t)\|^2 + |\theta'(t)|^2\Big), \tag{39}$$

$$-(\rho''(t),\theta'(t)) = -\int_0^1 \rho''(t)\theta'(t)\, dy \le \frac{1}{2}\Big(|\rho''(t)|^2 + |\theta'(t)|^2\Big), \tag{40}$$

$$\hat{a}(t,\rho,\theta') = \int_0^1 \Big(\frac{\partial a}{\partial y}(y,t)\Big)\Big(\frac{\partial \rho}{\partial y}\Big)\,\theta'\, dy \le c_5\Big(\|\rho(t)\|^2 + |\theta'(t)|^2\Big), \tag{41}$$

$$-\Big(b(y,t)\frac{\partial \rho'}{\partial y},\theta'\Big) = -\int_0^1 b(y,t)\Big(\frac{\partial \rho'}{\partial y}\Big)\theta'\, dy \le c_6\Big(\|\rho'(t)\|^2 + |\theta'(t)|^2\Big), \tag{42}$$

$$-\big(c(y,t)\frac{\partial \rho}{\partial y},\theta'\big) = -\int_0^1 c(y,t)\Big(\frac{\partial \rho}{\partial y}\Big)\theta'\, dy \le c_7\Big(\|\rho(t)\|^2 + |\theta'(t)|^2\Big). \tag{43}$$

Considering $\widetilde{c} = \max\{\frac{1}{2}, c_1, c_2, c_3, c_4, c_5, c_6, c_7\}$, a positive constant, and combining (38) through (43) with (35) we find

$$\begin{aligned}&\frac{1}{2}|\theta'(t)|^2 - \frac{1}{2}a(\xi,t)\|\theta(t)\|^2 \le \frac{1}{2}|\theta'(0)|^2 + \frac{1}{2}a(\xi,0)\ \|\theta(0)\|^2 \\ &+\widetilde{c}\int_0^t \Big\{\|\theta(s)\|^2 + |\theta'(s)|^2 + \|\rho(s)\|^2 + \|\rho'(s)\|^2 + |\rho''(s)|^2\Big\}\, ds.\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{1}{2}|\theta'(t)|^2 - \frac{1}{2}a(\xi,t)\|\theta(t)\|^2 \le \\ &\quad \frac{1}{2}|\theta'(0)|^2 + \frac{1}{2}a(\xi,0)\ \|\theta(0)\|^2 + \|\rho\|^2_{L^2((0,T);H_0^1(0,1))} + \|\rho'\|^2_{L^2((0,T);H_0^1(0,1))} \\ &\quad +|\rho''|^2_{L^2((0,T);L^2(0,1))} + \widetilde{c}\int_0^t \Big\{\|\theta(s)\|^2 + |\theta'(s)|^2\Big\}\, ds. \end{aligned} \tag{44}$$

Note that from hypotheses H1-H2 and definition (13), the function $a(y,t)$ is a strictly negative and bounded function on Q. Thus, taking $\hat{c} = \min\{\frac{1}{2}, -\frac{1}{2}a(\xi,t)\}$, we have $\hat{c} > 0$. Now, from (44) we obtain

$$|\theta'(t)|^2 + \|\theta(t)\|^2 \leq C_3\Big(|\theta'(0)|^2 + \|\theta(0)\|^2 + \|\rho\|^2_{L^2((0,T);H_0^1(0,1))} + \|\rho'\|^2_{L^2((0,T);H_0^1(0,1))} + |\rho''|^2_{L^2((0,T);L^2(0,1))}\Big) + C_4\int_0^T \Big(|\theta'(s)|^2 + \|\theta(s)\|^2\Big)ds, \quad (45)$$

where $C_3 = \dfrac{1}{2\hat{c}}\max\Big(1; a(\xi,0)\Big)$ and $C_4 = \dfrac{\widetilde{c}}{\hat{c}}$ are positive constants. From hypothesis H3 , we have the estimates,

$$\|\theta(0)\| \leq \overline{c}\, h; \qquad |\theta'(0)| \leq \widehat{c}\, h.$$

Moreover, from the usual finite element error estimates theory (see for instance [2]), and using the lemma of Douglas & Dupont [4], we can estimate the terms involving $\rho(t) = v(t) - \hat{v}_h(t)$ and its derivatives, namely

$$\|\rho\|_{L^2((0,T);H_0^1(0,1)\cap H^2(0,1))} \leq c_8 h, \qquad \|\rho'\|_{L^2((0,T);H_0^1(0,1))} \leq c_9 h,$$
$$|\rho''|_{L^2((0,T);L^2(0,1))} \leq c_{10} h.$$

Putting all these estimates together, there exists a positive constant C_5 such that

$$|\theta'(t)|^2 + \|\theta(t)\|^2 \leq C_5 h^2 + C_4\int_0^T \Big(|\theta'(s)|^2 + \|\theta(s)\|^2\Big)ds. \quad (46)$$

Taking $\varphi(t) = |\theta'(t)|^2 + \|\theta(t)\|^2 \geq 0$, considering (46) and applying the Gronwall lemma we conclude that

$$|\theta'(t)|^2 + \|\theta(t)\|^2 \leq C_6 h^2. \quad (47)$$

Since the right hand side of (47) is independent of t, that estimate implies immediately that

$$|\theta'|^2_{L^\infty((0,T);L^2(0,1))} + \|\theta\|^2_{L^\infty((0,T);H_0^1(0,1))} \leq C_6 h^2, \quad (48)$$

where the constant C_6 may take a positive value different from that in (47). Now, combining the error definition (22) and the interpolation results (24) with (48) and using the triangle inequality we arrive at

$$|e'|^2_{L^\infty((0,T);L^2(0,1))} + \|e\|^2_{L^\infty((0,T);H_0^1(0,1))} \leq C_7 h^2, \quad (49)$$

that is,

$$|v' - v_h'|_{L^\infty((0,T);L^2(0,1))} + \|v - v_h\|_{L^\infty((0,T);H_0^1(0,1))} \leq C_8 h. \quad (50)$$

where C_7 and C_8 are positive constants. □

4 Error Estimates of Problem (I)

Now let us restate the previous results for the original problem (I). For this we need one similar additional hypothesis as for Theorem 3, for the interpolators of the initial data, i.e,

$$\text{H3}': \qquad \|u_h(0) - \hat{u}_0\| \leq \overline{c}\, h; \qquad |u_h'(0) - \hat{u}_0'| \leq \widehat{c}\, h,$$

where $\hat{u}_0$ and $\hat{u}_0'$ are the interpolators of the initial conditions u_0, u_0', $\overline{c}$ and $\widehat{c}$ are positive constants independent h.

Theorem 4 *Let $\Omega_t = (\alpha(t), \beta(t))$, $\Omega_0 = (\alpha(0), \beta(0))$ and the initial data*

$$u_0 \in H_0^1(\Omega_0) \cap H^2(\Omega_0), \quad u_1 \in H_0^1(\Omega_0), \quad f, f' \in L^2((0,T); L^2(\Omega_t)),$$

for all $T > 0$. Under the hypotheses H1, H2 and H3′, the following estimate is valid,

$$|u' - u_h'|_{L^\infty((0,T);L^2(\Omega_t))} + \|u - u_h\|_{L^\infty((0,T);H_0^1(\Omega_t))} \leq Ch. \tag{51}$$

Proof. As mentioned before, the proof of Theorem 4 is established using the change of variables $\mathcal{T}$ defined in (7), i.e, if v is a solution of Theorem 3, then consider $u(x,t) = v(y,t)$, where $x = \alpha + \gamma y$. We also have $g(y,t) = f(x,t) = f(\alpha + \gamma y, t)$ and $v_0(y) = u(x,0) = u_0(\alpha(0) + \gamma(0)y)$, $v_1(y) = u'(x,0) = u_1(\alpha(0) + \gamma(0)y) + (\alpha'(0) + \gamma'(0)y)u_0'((\alpha(0) + \gamma(0)y)$. To verify that $u(x,t)$, under the hypotheses of Theorem 3, is a solution of problem (I), it is sufficient to observe that the mapping $\mathcal{T}^{-1}$ is of class C^2. Therefore, the operator $Lv(y,t)$ defined in (8), is transformed into the operator

$$\widetilde{L}u(x,t) = \frac{\partial^2 u}{\partial t^2} - \widetilde{a}(t)\frac{\partial^2 u}{\partial x^2},$$

with initial conditions u_0 and u_1. The error estimate of $v(y,t)$ given by (21) implies the error estimate of $u(x,t)$ in (51). $\square$

5 Concluding Remarks

In this paper we have shown that the error of the approximate solution of semi-discrete problem with moving boundary in Sobolev space is of the order of h, the maximum diameter of elements. This is an important result, since to our knowledge, besides existence, uniqueness and stability ([3, 5, 9, 10, 11, 12]) as well as numerical simulations ([8]) of vibration problem with moving boundary, error estimate of the approximate solution has not been analyzed.

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