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Original Article

Iterative approximation of stationary heat conduction in extended thermodynamics

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Boundary value problems for heat conduction treated by the 14-moment method of extended thermodynamics are considered. Iterated values for a physically uncontrollable boundary value are calculated and they converge quickly. Fields of temperature for the Fourier theory and for extended thermodynamics of 13 and 14 moments are compared.

1 Introduction

Boundary values present a problem in extended thermodynamics, because all we can control at the boundary are velocities, stresses, temperature and heat flux. Also, if we control the temperature at a part of the boundary we cannot control the heat flux there. And certainly we cannot control boundary values of higher moments like the 14^{th} moment.

One-dimensional heat conduction in the 14-moment theory is the simplest case in point: Here we have just one uncontrollable boundary value. It has been suggested in [1] that the body itself adjusts that value in such a manner as to minimize its entropy production. This assumption became known as the minimax principle. That principle is capable of determining the missing boundary conditions but it leads, to temperature fields that run counter to physical intuition and, more importantly, have never been observed.

Therefore we propose another criterion in this paper. We recall the iterative procedure for moment equations invented by Maxwell, and we employ an iteration of a similar spirit. There appear approximate solutions which depend on the uncontrollable boundary value and so does the exact solution. We obtain iterated values for the boundary value by requiring that – in each iterative step – the approximate solution is close to the exact one.

It turns out that the iterated boundary values converge quickly, both in the planar and cylindrically symmetric cases. Also the temperature fields do not exhibit the counter-intuitive features that we observed as a result of the minimax principle.

The resulting field is calculated and compared with the fields predicted by the Fourier theory and by extended thermodynamics of 13 moments.

The quest for boundary values has thrown extended thermodynamics into a state of flux. Thus it appears now that the 14-moment theory is not a good stepping stone on the road to more and more moments, if consistency in order of magnitude of non-equilibrium variables is required. However that aspect is ignored here. We refer the reader to the forthcoming paper [2]. Anyway, the present considerations about boundary values are unaffected by this new development. The inconsistencies of the 14-moment theory have prompted Brini & Ruggeri [3] to use the entropy principle for assigning different orders of magnitude to different non-equilibrium quantities and thus avoid the difficulties. Also those authors have extrapolated their approach to postulate values for uncontrollable boundary conditions. Their reasoning differs both from the present one and the one proposed in [2].

2 Field equations for 14 moments

The kinetic theory of gases defines moments of the distribution function in the form[1]

$$\rho^{i_1 i_2 \cdots i_n} = \int m\, c^{i_1} c^{i_2} \cdots c^{i_n} f\, d\boldsymbol{c}. \tag{2.1}$$

$f\, d\boldsymbol{c}$ is the number density of atoms of mass m at position $\boldsymbol{x}$ and time t which have the velocity $\boldsymbol{c}$. The first 13 moments are the fields of thermodynamics, viz.,

$$\begin{aligned} &\text{mass density} && \rho = \int m f\, d\boldsymbol{c}, \\ &\text{momentum density} && \rho^i = \rho\, v^i = \int m\, c^i f\, d\boldsymbol{c}, \\ &\text{pressure tensor} && \rho^{ij} = p\, g^{ij} - t^{\langle ij \rangle} = \int m\, c^i c^j f\, d\boldsymbol{c}, \\ &\text{heat flux} && \rho^{ij}{}_j = 2\, q^i = \int m\, c^2 c^i f\, d\boldsymbol{c}. \end{aligned} \tag{2.2}$$

v^i is the velocity of the gas which is taken to be zero throughout this paper since we are interested in heat conduction problems in a gas at rest. p is the pressure, $t^{\langle ij \rangle}$ the deviatoric stress and q^i the heat flux.

The more moments are specified the better is the description of the gas. A step in that direction is the inclusion of a 14$^{\text{th}}$ moment, viz.,

$$\rho^i{}_i{}^j{}_j = \int m\, c^4 f\, d\boldsymbol{c}. \tag{2.3}$$

The moment equations follow from the Boltzmann equation and if we allow for collisions by a BGK ansatz [4], they read

$$\frac{\partial \rho^{i_1 i_2 \cdots i_n}}{\partial t} + \rho^{i_1 i_2 \cdots i_n j}{}_{,j} = \frac{1}{\tau}\Big(\rho_E^{i_1 i_2 \cdots i_n} - \rho^{i_1 i_2 \cdots i_n}\Big). \tag{2.4}$$

$\rho_E^{i_1 i_2 \cdots i_n}$ is the equilibrium value of the moment of rank n which may be calculated from the Maxwell distribution

$$f_E = \frac{\rho}{m}\Big(2\pi \frac{k}{m}\theta\Big)^{-3/2} \exp\Big(-\frac{c^2}{2\frac{k}{m}\theta}\Big), \tag{2.5}$$

where k is the Boltzmann constant and θ is the kinetic temperature, a measure for the mean kinetic energy of the atoms. τ is a relaxation time of the order of magnitude of the mean time of free flight. It is assumed constant here.

In the case of the 14 moments (2.2), (2.3) we need to close the system (2.4) of moment equations by constitutive equations for

$$\rho^{ijk}, \quad \rho^i{}_i{}^{jk}, \quad \text{and} \quad \rho^i{}_i{}^j{}_j{}^k.$$

These quantities are related to the fields (2.2), (2.3) either by the Grad 14–moment distribution (cf. [5]), or by extended thermodynamics (cf. [6]). Restricting the attention to linear relations in non-equilibrium quantities we obtain

$$\begin{aligned} \rho^{ijk} &= \frac{2}{5}\Big(q^i g^{jk} + q^j g^{ki} + q^k g^{ij}\Big), \\ \rho^i{}_i{}^{jk} &= \frac{1}{3}\Big(15\,\frac{k}{m} p\theta + \varDelta\Big) g^{jk} - 7\,\frac{k}{m}\theta\, t^{\langle jk \rangle}, \\ \rho^i{}_i{}^j{}_j{}^k &= 28\,\frac{k}{m}\theta\, q^k, \end{aligned} \tag{2.6}$$

where $\varDelta$ stands for the non-equilibrium part of the 14$^{\text{th}}$ moment.

Insertion of the constitutive equations (2.6) into the 14-moment equations provides a set of field equations for the fields (2.2), (2.3), which we proceed to solve in a simple case.

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[1] We employ the usual tensor notation with upper and lower indices as contravariant and covariant components respectively. g^{ij} are the contravariant components of the metric tensor. A comma denotes a covariant derivative. In Cartesian coordinates, we use lower indices throughout.

The density of entropy production in a 14–moment theory has the form, e.g., see [6]

$$\sigma = \frac{1}{\tau}\Big(\frac{1}{2}\,\frac{t^{\langle ij\rangle}t_{\langle ij\rangle}}{p\,\theta} + \frac{2}{5}\,\frac{q^i q_i}{\frac{k}{m}p\,\theta^2} + \frac{1}{120}\,\frac{\Delta^2}{(\frac{k}{m})^2 p\,\theta^3}\Big), \tag{2.7}$$

which is a non-negative quadratic form in the non-equilibrium parts of the state variables. We shall have occasion to return to this expression.

3 Stationary heat conduction

We consider a gas at rest between two parallel plates. In that case Cartesian coordinates are appropriate and the 14-moment system assumes the form,

$$\begin{aligned}
&p_{,j} - t_{\langle jk\rangle,k} = 0,\\
&\frac{2}{5}(q_{i,j} + q_{j,i} + q_{k,k}\delta_{ij}) = \frac{1}{\tau}\,t_{\langle ij\rangle},\\
&\frac{1}{3}\Delta_{,j} + 5\frac{k}{m}(p\,\theta)_{,j} - 7\frac{k}{m}(\theta\,t_{\langle jk\rangle})_{,k} = -\frac{2}{\tau}\,q_j,\\
&28\,\frac{k}{m}(\theta q_k)_{,k} = -\frac{1}{\tau}\Delta.
\end{aligned} \tag{3.1}$$

3.1 Two boundary value problems

Let the gas rest between plates at $x_1 = 0$ and $x_1 = L$. We look at a process in which all fields depend on $x = x_1$ only and where the heat flux is in the x–direction. In this case we have, from the first two equations of (3.1),

$$\frac{dp}{dx} = 0, \qquad \frac{dq}{dx} = 0, \qquad t_{\langle ij\rangle} = 0,$$

and the remaining two equations reduce to

$$\begin{aligned}
5\frac{k}{m}p\,\frac{d\theta}{dx} + \frac{1}{3}\,\frac{d\Delta}{dx} &= -\frac{2}{\tau}\,q,\\
28\frac{k}{m}\,\frac{d(\theta q)}{dx} &= -\frac{1}{\tau}\Delta.
\end{aligned} \tag{3.2}$$

We conclude that the pressure and the heat flux are constants. Also the system (3.2) can easily be solved.

Prescribing boundary values $\theta(0)$, $q(L)$ and $\Delta(0)$, we obtain the solution,

$$\begin{aligned}
\theta &= \theta(0) - \frac{2}{5}\,\frac{q(L)}{\frac{k}{m}\tau p}\,x - \frac{1}{15\frac{k}{m}p}\Big(\Delta(0) - \frac{56}{5}\,\frac{q(L)^2}{p}\Big)\exp\Big(\frac{15}{28}\frac{1}{\tau}\,\frac{p}{q(L)}\,x\Big) - 1,\\
\Delta &= \frac{56}{5}\,\frac{q(L)^2}{p} + \Big(\Delta(0) - \frac{56}{5}\,\frac{q(L)^2}{p}\Big)\exp\Big(\frac{15}{28}\frac{1}{\tau}\,\frac{p}{q(L)}\,x\Big).
\end{aligned} \tag{3.3}$$

If however we prescribe the boundary values $\theta(0)$, $\theta(L)$ and $\Delta(0)$, we obtain the same solution, except that $q(L)$ must now be determined from the equation,

$$\theta(L) - \theta(0) + \frac{2}{5}\,\frac{L}{\tau\frac{k}{m}p}q(L) + \frac{1}{15\frac{k}{m}p}\Big(\Delta(0) - \frac{56}{5}\,\frac{q(L)^2}{p}\Big)\exp\Big(\frac{15}{28}\frac{L}{\tau}\,\frac{p}{q(L)}\Big) - 1 = 0; \tag{3.4}$$

thus $q(L)$ is a function of the boundary values $\theta(0)$, $\theta(L)$ and $\Delta(0)$. The pressure p is a given constant.

In the classical Fourier theory of heat conduction in a gas at rest a problem is well-posed if

$$\text{either} \qquad \theta(0), q(L) \qquad \text{or} \qquad \theta(0), \theta(L)$$

are prescribed. These problems have a unique and physically meaningful solution, which may be obtained from (3.3) and (3.4) by setting $\Delta(0) = \frac{56}{5}\,q(L)^2/p$. However, for extended thermodynamics we generally need additional boundary values for higher moments to determine the solution. Thus, by (3.3) the 14-moment theory requires a boundary value for the non-equilibrium value Δ of the 14$^{\text{th}}$ moment. And that is where the problem lies, because $\Delta(0)$ cannot be prescribed. We refer to this quantity as an uncontrollable boundary value.

Since, by (3.2)$_2$, Δ is related to the temperature gradient, we may say that the 14-moment theory requires boundary values for θ and $d\theta/dx$ at $x = 0$, and – of course – either q or θ at $x = L$. However that does not help. Indeed, the only controllable boundary values are those of the velocities, the stresses, temperature and heat flux. While mathematically gradients – of temperature (say) – may be prescribed, this is physically impractical or, in fact, impossible.

Therefore the 14-moment theory cannot be regarded as a useful improvement of the classical Fourier theory, *unless* we find a criterion from which the necessary additional boundary value can be determined. It has been conjectured that such a criterion may be based on an *a priori* assumption concerning the entropy production.

3.2 Minimum entropy production

The entropy production (2.7) is a non-negative quadratic function in $(t^{\langle ij\rangle}, q^j, \Delta)$ and the coefficients depend on p and θ. Synthetically, we may write

$$\sigma = \sigma(u, X), \quad \text{for} \quad X = (t^{\langle ij\rangle}, q^j, \Delta) \quad \text{and} \quad u = (p, \theta).$$

We may define the L^p norm of σ by

$$\|\sigma(u,X)\|_p = \begin{cases} \left\{ \int_0^L \sigma(u(x), X(x))^p \, dx \right\}^{1/p} & \text{for } 1 \leq p < \infty, \\ \max\limits_{1 \leq x \leq L} \sigma(u(x), X(x)) & \text{for } p = \infty. \end{cases} \tag{3.5}$$

This norm provides a convenient measure for the deviation of a process $(u(x), X(x))$ from the equilibrium. And since, by (3.3), we have

$$\theta = \theta(x, \Delta(0)) \qquad \text{and} \qquad \Delta = \Delta(x, \Delta(0)),$$

the norm $\|\sigma(u,X)\|_p$ depends on $\Delta(0)$ as a parameter.

Struchtrup and Weiss [1] have proposed that the gas selects *the* value $\Delta(0)$ which has the smallest entropy production in the L^∞ norm. In other words, they have determined the value of $\Delta(0)$ by requiring that

$$\|\sigma(u, X; \Delta(0))\|_p \leq \ \|\sigma(u, X; \alpha)\|_p \quad \forall\, \alpha \in I\!R \quad \text{with} \quad p = \infty. \tag{3.6}$$

This postulate has become known as the *Minimax Principle*. The proposition traces back to the *Principle of Minimum Entropy Production* of Prigogine [7], who postulated that the global entropy production assumes a minimum in a stationary process.[2] For the present problem the Prigogine principle corresponds to the criterion (3.6) with $p = 1$, instead of $p = \infty$. In [1] it is shown that the principle does not provide reasonable results for our problem of heat conduction.

However, the minimax principle also provides strange results. To show this we investigate the heat conduction problems solved in Paragraph 3.1. We consider argon gas and choose $L = 0.01\,\text{m}$, $\tau = 10^{-5}\,\text{sec}$, $p = 100\,\text{Pa}$, and the boundary conditions,

$$\theta(0) = 300\,\text{K} \quad \text{and} \quad q(L) = -5000\,\text{Wm}^{-2}.$$

We apply the minimax principle to determine $\Delta(0)$ and obtain

$$\Delta(0) = -1.835\text{x}10^4\text{Ns}^{-2} \quad \text{and hence by (3.3)}_1 \quad \theta(L) = 387.25\,\text{K}.$$

On the other hand, if we choose boundary conditions as

$$\theta(0) = 300\,\text{K} \quad \text{and} \quad \theta(L) = 387.25\,\text{K},$$

the minimax principle provides

$$\Delta(0) = 2.674\text{x}10^6\text{Ns}^{-2} \quad \text{and hence by (3.4)} \quad q(L) = -4455.00\,\text{Wm}^{-2}.$$

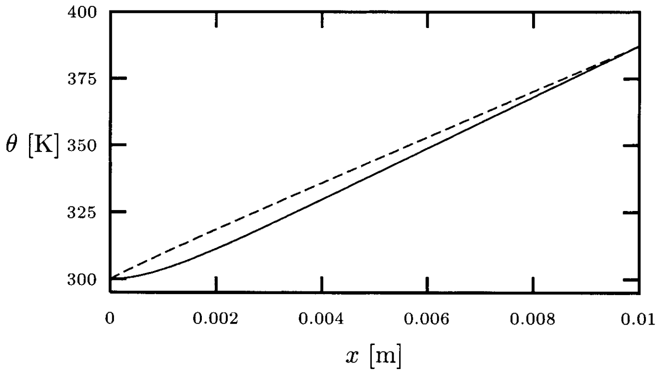

Fig. 1. Temperature $\theta(x)$ by Minimax Principle with boundary conditions: $(\theta(0), q(L))$ – solid line, and $(\theta(0), \theta(L))$ – dashed line

The two solutions are shown in Fig. 1 and we see that they differ considerably. Of course, they *are* different problems, so that different solutions should not surprise us. Yet, our physical intuition says, that for the same boundary temperatures a sensible criterion ought to select the same value $\varDelta(0)$, and the minimax principle does not do that; we shall refer to this phenomenon as lack of consistency between the solutions. [Incidentally, the case considered in [1] corresponds to the dashed line in Fig. 1 whose slope is almost constant as should be expected in the Fourier theory. This observation has originally inspired confidence in the minimax principle and has led to the studies [9,10] of similar problems].

3.3 Consistency requirement

We wish to avoid the above described inconsistency and we shall forbid it. In fact we make consistency a crucial test for the validity of solutions of boundary value problems.

Once we have a criterion for the determination of the proper value $\varDelta(0)$, we shall test the criterion by the

Consistency requirement: The uncontrollable boundary value $\varDelta(0)$, however it is selected, has to be such that the solutions of the two boundary value problems

(I): with boundary conditions, $\theta^{\mathrm{I}}(0)$, $q^{\mathrm{I}}(L)$, $\varDelta^{\mathrm{I}}(0)$ and hence $\theta^{\mathrm{I}}(L)$

(II): with boundary conditions, $\theta^{\mathrm{I}}(0)$, $\theta^{\mathrm{I}}(L)$, $\varDelta^{\mathrm{II}}(0)$ and hence $q^{\mathrm{II}}(L)$

are the same solutions. In particular, we must have $\varDelta^{\mathrm{I}}(0) = \varDelta^{\mathrm{II}}(0)$ and $q^{\mathrm{I}}(L) = q^{\mathrm{II}}(L)$.

It remains to find the criterion for the determination of $\varDelta(0)$. We shall rely on iterated solutions for this purpose.

4 Iterative approximation

Recall that the Maxwellian iteration (see [11]) may be regarded as a method to obtain approximate constitutive relations in ordinary thermodynamics from the additional balance equations in extended thermodynamics. We shall be guided by this idea in order to determine the proper boundary value $\varDelta(0)$ in our extended 14-moment theory of heat conduction.

4.1 The iterative scheme

What we employ as an iterative scheme is not the Maxwellian iteration, but it is somewhat similar to it. We use the balance equations (3.2) to define n^{th} iterates for q and $\varDelta$ in terms of $(n-1)^{\mathrm{th}}$ iterates via the recipe,

$$\begin{aligned} \overset{(n)}{q} &= -\frac{5}{2}\,\tau p \frac{k}{m}\,\frac{d\theta}{dx} - \frac{1}{6}\,\tau\,\frac{d}{dx}(\overset{(n\text{-}1)}{\varDelta}), \\ \overset{(n)}{\varDelta} &= -28\,\tau \frac{k}{m}\,\frac{d}{dx}(\theta\,\overset{(n\text{-}1)}{q}), \end{aligned} \tag{4.1}$$

[2] This principle is valid only for some very restricted cases as pointed out in [8].

for $n = 1, 2, 3, \cdots$, where $\theta = \theta(x, \varDelta(0))$ is the solution for the temperature given in $(3.3)_1$ as

$$\theta = \theta(0) - \frac{q(L)}{\kappa} x - \frac{1}{15\frac{k}{m}p}(\varDelta(0) - D)(e^{\gamma x} - 1). \tag{4.2}$$

The constant D, γ and κ may be identified by comparison with $(3.3)_1$, viz.

$$D = \frac{56}{5}\frac{q(L)^2}{p}, \qquad \gamma = \frac{15}{28}\frac{1}{\tau}\frac{p}{q(L)}, \qquad \kappa = \frac{5}{2}\frac{k}{m}\tau p.$$

For brevity we define

$$\overset{(n)}{X}(x, \varDelta(0)) = \big(\overset{(n)}{q}(x, \varDelta(0)), \overset{(n)}{\varDelta}(x, \varDelta(0))\big)$$

and use

$$\overset{(0)}{X}: \quad \overset{(0)}{q} = 0, \quad \overset{(0)}{\varDelta} = 0,$$

as the initiation agreement. The first four iterates are thus given by

$$\overset{(1)}{X}: \quad \overset{(1)}{\varDelta} = 0\,, \qquad \overset{(1)}{q} = -\frac{5}{2}\tau\frac{k}{m}p\,\theta'\,;$$

$$\overset{(2)}{X}: \quad \overset{(2)}{q} = \overset{(1)}{q}\,, \qquad \overset{(2)}{\varDelta} = 70\,\tau^2(\frac{k}{m})^2 p\,(\theta'^2 + \theta\,\theta'')\,;$$

$$\overset{(3)}{X}: \quad \overset{(3)}{\varDelta} = \overset{(2)}{\varDelta}\,, \qquad \overset{(3)}{q} = \overset{(2)}{q} - \frac{35}{3}\tau^3(\frac{k}{m})^2 p\,(3\,\theta'\,\theta'' + \theta\,\theta''')\,;$$

$$\overset{(4)}{X}: \quad \overset{(4)}{q} = \overset{(3)}{q}\,, \qquad \overset{(4)}{\varDelta} = \overset{(3)}{\varDelta} + \frac{980}{3}\tau^4(\frac{k}{m})^3 p\,(3\,\theta'^2\theta'' + 3\,\theta\,\theta''^2 + 5\,\theta\,\theta'\theta''' + \theta^2\theta'''')\,.$$

In these relations, we have denoted the gradient of θ by θ' for simplicity.

All these iterates depend explicitly on $\varDelta(0)$, because $\theta(x)$, by (4.2), depends on $\varDelta(0)$.

4.2 Iterative minimization

Of course, the *exact* solution $X(x, \varDelta(0)) = \big(q(x, \varDelta(0)), \varDelta(x, \varDelta(0))\big)$ of the field equations (3.1) reads

$$X: \quad q(x) = q(L), \quad \varDelta(x) = D + (\varDelta(0) - D)e^{\gamma x}. \tag{4.3}$$

This exact solution X and all the iterates $\overset{(n)}{X}$ depend on the uncontrollable boundary value $\varDelta(0)$. Since this value cannot be prescribed we must suppose that the gas itself chooses the proper value which we shall denote by $\bar{\varDelta}(0)$. We identify iterates $\bar{\varDelta}_n(0)$ of this proper value by the following postulate of

Iterative minimization: We find the value of $\bar{\varDelta}_n(0)$ in each iterate step by requiring that the iterated solution $\overset{(n)}{X}(x, \varDelta(0))$ is as close as possible to the exact solution $X(x, \varDelta(0))$.

Since we expect the iterates $\overset{(n)}{X}$ to be approximations to the exact solution we expect the values $\bar{\varDelta}_n(0)$ to converge as $n \to \infty$. Also we expect the consistency requirement of Paragraph 3.3 to be better satisfied when n increases.

As a measure of closeness of the solutions X and $\overset{(n)}{X}$ we choose a norm based on the entropy production density (2.7) but for a constant reference temperature $\tilde{\theta}$,

$$\|X\|_\sigma = \Big\{\int_0^L \sigma(X(x))\,dx\Big\}^{1/2}, \tag{4.4}$$

where $X = (q, \Delta)$ and

$$\sigma(q,\Delta) = \frac{2}{5\tau \frac{k}{m} p\,\tilde{\theta}^2}\, q^2 + \frac{1}{120\tau(\frac{k}{m})^2 p\,\tilde{\theta}^3}\, \Delta^2. \tag{4.5}$$

It is equivalent to the L^2 norm in the space of $X = (q, \Delta) \in I\!R^2$. This norm is chosen because q and Δ are different physical quantities, and to account for consistency in physical units, proper coefficients for q^2 and Δ^2 are taken from the expression of the entropy production density.[3]

Therefore, we calculate the iterates $\bar{\Delta}_n(0)$ from the requirement

$$\left\| X(\bar{\Delta}_n(0)) - \overset{(n)}{X}(\bar{\Delta}_n(0)) \right\|_\sigma \le \left\| X(\Delta(0)) - \overset{(n)}{X}(\Delta(0)) \right\|_\sigma \qquad \forall\, \Delta(0) \in I\!R. \tag{4.6}$$

For the same data as before, numerical minimizations for the first few iterative steps give the following results:

$$\bar{\Delta}_1(0) = 2.267\text{x}10^6\text{Ns}^{-2}, \quad \bar{\Delta}_2(0) = 2.8\text{x}10^6\text{Ns}^{-2}, \quad \bar{\Delta}_3(0) = 2.8\text{x}10^6\text{Ns}^{-2}, \quad \bar{\Delta}_4(0) = 2.8\text{x}10^6\text{Ns}^{-2}.$$

In fact, we can also show analytically that the norm $\left\| X(\Delta(0)) - \overset{(n)}{X}(\Delta(0)) \right\|_\sigma$ contains a factor $(\Delta(0) - D)$, for $n \ge 2$, so that the minimum value is attained at $\Delta(0) = D$. In other words, the sequence $\{\bar{\Delta}_1(0), \bar{\Delta}_2(0), \cdots\}$ converges in a trivial way at the second step and we obtain the proper value $\bar{\Delta}(0)$ as

$$\bar{\Delta}(0) = \lim_{n\to\infty} \bar{\Delta}_n(0) = D = \frac{56}{5}\,\frac{q(L)^2}{p} = 2.8\text{x}10^6\text{Ns}^{-2}.$$

Consequently, we have the solution of the boundary value problem for (3.1) in the form

$$q(x) = q(L)\,, \qquad \Delta(x) = \frac{56}{5}\,\frac{q(L)^2}{p}\,, \qquad \theta(x) = \theta(0) - \frac{q(L)}{\frac{5}{2}\frac{k}{m}\tau p}\, x\,.$$

In this case, the consistency requirement of Paragraph 3.3 is trivially satisfied for all $n \ge 2$. Indeed, since $\Delta(x)$ is constant and $\theta(x)$ is independent of $\Delta(0)$, the problem is reduced to the classical one of Fourier heat conduction. This case is too simple to appreciate the test of consistency. Therefore in Sect. 5 we proceed to consider the more complex case of cylindrically symmetric heat conduction.

Before that, however, we investigate some properties of the iterative approximation.

4.3 A remark on the convergence of the iterative approximation

Despite the simplicity of the case of planar heat conduction, it is instructive to study the convergence of the iterates $\overset{(n)}{X}(\Delta(0))$.

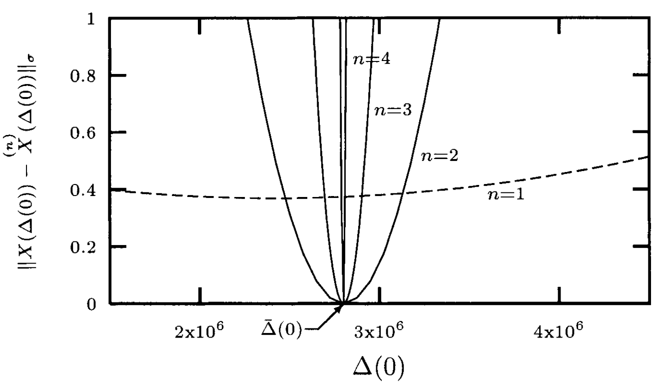

Fig. 2. Dependence of σ–norm on $\Delta(0)$

In Fig. 2, we have plotted the norms $\| X(\Delta(0)) - \overset{(n)}{X}(\Delta(0)) \|_\sigma$ for different values of n as functions of $\Delta(0)$. We observe that the curves are convex so that a unique minimum exists. Also inspection shows that the intervals

$$I_n = \{\Delta(0) \in I\!R : \| X(\Delta(0)) - \overset{(n)}{X}(\Delta(0)) \|_\sigma \le M\}$$

[3] This choice of norm is convenient; it does not represent yet another principle of minimum entropy production. In fact, the σ in the definition is *not* the entropy production, because of the constant reference temperature $\tilde{\theta}$.

are nested for a given $M > 0$, so that we have

$$I_1 \supset I_2 \supset I_3 \supset I_4.$$

Therefore we expect

$$I_n \supset I_{n+1} \qquad \text{for } n = 1, 2, \cdots .$$

If this is true, there is a common interval

$$I_\infty = \bigcap_{n=1}^{\infty} I_n,$$

and, by (4.6), the limit $\bar{\Delta}(0)$, if it exists, must lie in that interval.

We may rephrase that conclusion as follows: For a given $M > 0$, if $\Delta(0) \notin I_\infty$, then there exists an $N > 0$ such that

$$\|X(\Delta(0)) - \overset{(n)}{X}(\Delta(0))\|_\sigma > M \qquad \text{for } n > N.$$

Therefore, we conclude that the iterates $\overset{(n)}{X}(\Delta(0))$ do not converge to the solution $X(\Delta(0))$ of the moment equations for any $\Delta(0) \notin I_\infty$.

In other words: $\Delta(0)$ cannot be assigned arbitrarily if the iterates are to be approximations of the exact solution.

We might even conjecture that the proper uncontrollable boundary value $\bar{\Delta}(0)$ may be determined as *the one* that permits convergence of our iteration. However this has not been proven, because the higher iterates are too complicated for proving convergence of the interval I_∞ to a single point $\{\bar{\Delta}(0)\}$.

5 Heat conduction between coaxial cylinders

Insertion of the constitutive equations (2.6) into the moment equations (2.4) leads to the following explicit system of field equations

$$\begin{aligned}
& g^{jk}\frac{\partial p}{\partial x^k} - \frac{\partial t^{\langle jk\rangle}}{\partial x^k} - \Gamma_{k\ i}^{\ j} t^{\langle ik\rangle} - \Gamma_{k\ i}^{\ k} t^{\langle ji\rangle} = 0, \\
& \frac{\partial q^k}{\partial x^k} + \Gamma_{k\ i}^{\ k} q^i = 0, \\
& \frac{2}{5}\Big(g^{ik}\frac{\partial q^j}{\partial x^k} + g^{jk}\frac{\partial q^i}{\partial x^k} + g^{ik}\Gamma_{k\ l}^{\ j} q^l + g^{jk}\Gamma_{k\ l}^{\ i} q^l\Big) = \frac{1}{\tau}\, t^{\langle ij\rangle}, \\
& 5\frac{k}{m} g^{jk}\frac{\partial (p\,\theta)}{\partial x^k} + \frac{1}{3} g^{jk}\frac{\partial \Delta}{\partial x^k} - 7\frac{k}{m}\, t^{\langle jk\rangle}\frac{\partial \theta}{\partial x^k} - 7\frac{k}{m}\theta\Big(\frac{\partial t^{\langle jk\rangle}}{\partial x^k} + \Gamma_{k\ i}^{\ j} t^{\langle ik\rangle} + \Gamma_{k\ i}^{\ k} t^{\langle ji\rangle}\Big) = -\frac{2}{\tau}\, q^j, \\
& 28\,\frac{k}{m} q^k\frac{\partial \theta}{\partial x^k} + 28\,\frac{k}{m}\theta\Big(\frac{\partial q^k}{\partial x^k} + \Gamma_{k\ i}^{\ k} q^i\Big) = -\frac{1}{\tau}\,\Delta\,.
\end{aligned} \tag{5.1}$$

The cylindrical coordinates – $(x^1, x^2, x^3) = (r, \vartheta, z)$ – will be used with the metric tensor and the Christoffel symbols given by

$$[g^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Gamma_{\vartheta\ \vartheta}^{\ r} = -r, \ \ \Gamma_{\vartheta\ r}^{\ \vartheta} = \Gamma_{r\ \vartheta}^{\ \vartheta} = \frac{1}{r}, \ \ \Gamma_{i\ k}^{\ j} = 0 \ \text{ else.}$$

5.1 Solutions

We consider a gas at rest between two coaxial cylinders. We assume a stationary process in which all fields depend only on the radial coordinate r, and the heat flux has only a radial component, with $q^r = q(r)$ and

$q^{\vartheta} = 0$, $q^{z} = 0$. In this case, from (5.1) a little calculation leads to the following field equations,

$$\begin{aligned}
&\frac{dp}{dr} = 0, \qquad \frac{dq}{dr} + \frac{1}{r}\,q = 0,\\
&t^{\langle rr\rangle} = \frac{4}{5}\,\tau\,\frac{dq}{dr}, \qquad t^{\langle\vartheta\vartheta\rangle} = \frac{4}{5}\,\tau\frac{1}{r^3}\,q,\\
&q = -\frac{5}{2}\,\tau\frac{k}{m}p\,\frac{d\theta}{dr} - \frac{1}{6}\,\tau\frac{d\Delta}{dr} + \frac{7}{2}\,\tau\frac{k}{m}\,t^{\langle rr\rangle}\frac{d\theta}{dr},\\
&\Delta = -28\tau\frac{k}{m}q\,\frac{d\theta}{dr}\,.
\end{aligned} \tag{5.2}$$

In this calculation, we have noted that all other components of $t^{\langle ij\rangle}$ are zero and $t^{\langle jk\rangle}{}_{,k} = 0$, i.e.,

$$\frac{\partial t^{\langle jk\rangle}}{\partial x^k} + \Gamma^{\,j}_{k\,i}t^{\langle ik\rangle} + \Gamma^{\,k}_{k\,i}t^{\langle ji\rangle} = 0.$$

Therefore, the pressure p is constant and

$$q = \frac{c}{r}, \qquad t^{\langle rr\rangle} = -\frac{4}{5}\,\frac{\tau c}{r^2}, \qquad t^{\langle\vartheta\vartheta\rangle} = \frac{4}{5}\,\frac{\tau c}{r^4}, \tag{5.3}$$

while the solutions for the temperature θ and the non-equilibrium moment Δ can be determined from the last two equations in (5.2) numerically. In this solution, besides, the constants k/m, τ and p, considered as given, three integration constants occur.

The constants of integration may be determined from two sets of boundary values at the inner and outer cylinders with radii r_{i} and r_{o} respectively,

(I): $\theta(r_{\mathrm{o}}) = \theta_{\mathrm{o}}$, $q(r_{\mathrm{i}}) = q_{\mathrm{i}}$, $\Delta(r_{\mathrm{o}}) = \Delta_{\mathrm{o}}$,

(II): $\theta(r_{\mathrm{o}}) = \theta_{\mathrm{o}}$, $\theta(r_{\mathrm{i}}) = \theta_{\mathrm{i}}$, $\Delta(r_{\mathrm{o}}) = \Delta_{\mathrm{o}}$.

In both triples we have two controllable boundary values but also the uncontrollable one Δ_{o}. In particular, $\theta(r)$ and $\Delta(r)$ depend parametrically on Δ_{o},

$$\theta = \theta(r, \Delta_{\mathrm{o}}), \qquad \Delta = \Delta(r, \Delta_{\mathrm{o}}).$$

The uncontrollable boundary value Δ_{o} will be determined by the iterative approximation in a manner analogous to Sect. 4.

5.2 Iterative approximation

We combine the exact solution of the non-equilibrium fields by

$$X(r, \Delta_{\mathrm{o}}) = \big(t^{\langle rr\rangle}(r), t^{\langle\vartheta\vartheta\rangle}(r), q(r), \Delta(r, \Delta_{\mathrm{o}})\big),$$

and the n^{th} iterate by

$$\overset{(n)}{X}(r, \Delta_{\mathrm{o}}) = \big(\overset{(n)}{t}{}^{\langle rr\rangle}(r, \Delta_{\mathrm{o}}), \overset{(n)}{t}{}^{\langle\vartheta\vartheta\rangle}(r, \Delta_{\mathrm{o}}), \overset{(n)}{q}(r, \Delta_{\mathrm{o}}), \overset{(n)}{\Delta}(r, \Delta_{\mathrm{o}})\big).$$

From (5.2) the following iterative scheme suggests itself,

$$\begin{aligned}
&\overset{(n)}{t}{}^{\langle rr\rangle} = \frac{4}{5}\tau\,\frac{d}{dr}(\overset{(n-1)}{q}), \qquad \overset{(n)}{t}{}^{\langle\vartheta\vartheta\rangle} = \frac{4}{5}\tau\frac{1}{r^3}\,\overset{(n-1)}{q},\\
&\overset{(n)}{q} = -\frac{5}{2}\,\tau\frac{k}{m}p\,\frac{d\theta}{dr} - \frac{1}{6}\,\tau\frac{d}{dr}(\overset{(n-1)}{\Delta}) + \frac{7}{2}\,\tau\frac{k}{m}\,\frac{d\theta}{dr}\,\overset{(n-1)}{t}{}^{\langle rr\rangle},\\
&\overset{(n)}{\Delta} = -28\,\tau\frac{k}{m}\frac{d\theta}{dr}\,\overset{(n-1)}{q},
\end{aligned} \tag{5.4}$$

with the initiation agreement $\overset{(0)}{X} = (0, 0, 0, 0)$. Note that in the scheme (5.4) the temperature is given by the solution $\theta(r, \Delta_{\mathrm{o}})$. At each iterative step, we have the following problem of

Iterative minimization: Find the value $\bar{\varDelta}_{\rm o}^n$ in each step n by requiring that the iterated solution $\overset{(n)}{X}(r, \varDelta_{\rm o})$ is as close as possible to the exact solution $X(r, \varDelta_{\rm o})$. The norm

$$\|X\|_\sigma = \Big\{ \int_0^L \sigma(X(x))\, dx \Big\}^{1/2}, \qquad X = (t^{\langle rr\rangle}, t^{\langle\vartheta\vartheta\rangle}, q, \varDelta) \in I\!R^4,$$

which measures closeness is again based on the entropy production density, cf. (2.7), in the present case,

$$\sigma(t^{\langle rr\rangle}, t^{\langle\vartheta\vartheta\rangle}, q, \varDelta) = \frac{1}{2\tau p\,\tilde\theta}(t^{\langle rr\rangle}t_{\langle rr\rangle} + t^{\langle\vartheta\vartheta\rangle}t_{\langle\vartheta\vartheta\rangle}) + \frac{2}{5\tau\frac{k}{m}p\,\tilde\theta^2}\,q^2 + \frac{1}{120\tau(\frac{k}{m})^2p\,\tilde\theta^3}\,\varDelta^2,$$

where $\tilde\theta$ is a constant reference temperature. The minimization problem for the determination of $\bar{\varDelta}_{\rm o}^n$ reads

$$f(\bar{\varDelta}_{\rm o}^n) = \big\|X(\bar{\varDelta}_{\rm o}^n) - \overset{(n)}{X}(\bar{\varDelta}_{\rm o}^n)\big\|_\sigma \le \big\|X(\varDelta_{\rm o}) - \overset{(n)}{X}(\varDelta_{\rm o})\big\|_\sigma \qquad \forall\, \varDelta_{\rm o} \in I\!R. \tag{5.5}$$

5.3 Numerical results

Table 1 shows the numerical results of the iterative minimization for different iterative steps and for $r_{\rm i} = 0.01\,$m, $r_{\rm o} = 0.03\,$m, $\tau = 10^{-5}$ sec, $p = 100\,$Pa. The minimization is applied to

Problem (I) with $\theta_{\rm o}^{\rm I} = 300\,$K, $q_{\rm i}^{\rm I} = 10^4\,{\rm Wm}^{-2}$ and

Problem (II) with $\theta_{\rm o}^{\rm II} = 300\,$K, $\theta_{\rm i}^{\rm II} = \theta^{\rm I}(r_{\rm i})$.

Note that in Problem (II) an inner temperature is prescribed that is the result of the boundary value Problem (I). This is to check consistency in the sense of Paragraph 3.3 of course. Also in Problem (II) for consistency we must obtain $q^{\rm II}(r_{\rm i}) = 10^4\,{\rm Wm}^{-2}$, and we do obtain that value quite well, at least for $n \ge 2$.

Table 1. Results of the minimization

	Problem (I)			Problem (II)		
n	$\bar{\varDelta}_{\rm o}^n$	$\theta^{\rm I}(r_{\rm i})$	$f^{\rm I}(\bar{\varDelta}_{\rm o}^n)$	$\bar{\varDelta}_{\rm o}^n$	$q^{\rm II}(r_{\rm i})$	$f^{\rm II}(\bar{\varDelta}_{\rm o}^n)$
1	1.1943x10^6	482.8031	2.0594x10^{-1}	2.0383x10^6	9824.6	2.0292x10^{-1}
2	1.1848x10^6	482.7726	8.8811x10^{-2}	1.1980x10^6	9997.2	8.8789x10^{-2}
3	1.1749x10^6	482.7411	5.5844x10^{-2}	1.1805x10^6	9998.8	5.5837x10^{-2}
4	1.1825x10^6	482.7653	4.0850x10^{-2}	1.1841x10^6	9999.7	4.0849x10^{-2}
5	1.1833x10^6	482.7677	3.3185x10^{-2}	1.1843x10^6	9999.8	3.3184x10^{-2}

Also the required values $\bar{\varDelta}_{\rm o}^n$ converge quite rapidly with increasing n as the table shows. And not only that: The values of $\bar{\varDelta}_{\rm o}^n$ tend to the *same* limits for both problems, thus proving consistency of the iterative minimization for both problems. The columns for $f(\bar{\varDelta}_{\rm o}^n)$, cf. (5.5), provide an estimate for the quality of the approximation in each step.

Table 2. Consistency error

$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
2.3105%	0.03642%	0.01542%	0.004370%	0.002859%

The precision of the iterative approximation can also be estimated from the "consistency error", defined as

$$\frac{\|X(\bar{\varDelta}_{\rm o}^n(\textsc{i})) - X(\bar{\varDelta}_{\rm o}^n(\textsc{ii}))\|_\sigma}{\|X(\bar{\varDelta}_{\rm o}^n(\textsc{i}))\|_\sigma}.$$

This quantity is listed in Table 2 and inspection shows that the consistency of the two types of boundary value problems improves rapidly as n increases.

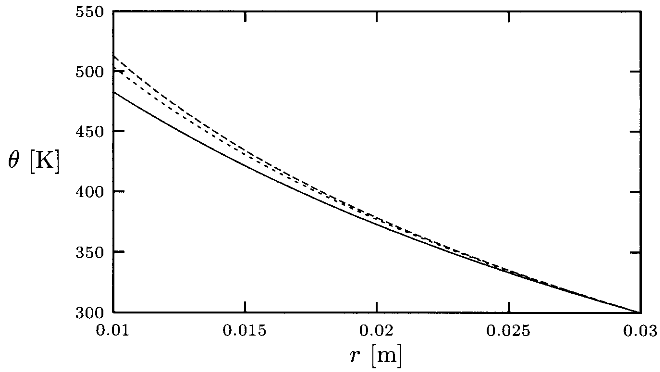

Fig. 3. Temperature $\theta(r)$ for Fourier theory (*upper*), 13-moment theory (*middle*) and 14-moment theory (lower).

In Fig. 3, the approximate solution for the temperature field $\theta(r)$, with the value $\bar{\Delta}_{\rm o}^{5} = 1.183\text{x}10^{6}$ obtained in the $5^{\rm th}$ iteration, is plotted in solid line. For comparison, the results from the Fourier theory and the 13-moment theory are also plotted. The Fourier solution is given by

$$\theta(r) = \theta_{\rm o} - \frac{q_{\rm i} r_{\rm i}}{\kappa} \log \frac{r}{r_{\rm o}}, \quad \text{where} \quad \kappa = \frac{5}{2}\frac{k}{m}\tau p.$$

The corresponding problem in the 13-moment theory does not require an additional boundary condition and the solution is given by

$$\theta(r) = \theta_{\rm o} - \frac{q_{\rm i} r_{\rm i}}{2\kappa} \log \frac{r^2 + b\, q_{\rm i} r_{\rm i}}{r_{\rm o}^2 + b\, q_{\rm i} r_{\rm i}}, \quad \text{where} \quad b = \frac{28}{25}\frac{\tau}{p}.$$

This result has been reported in [12].

We note that the extended theories differ and that both differ from the Fourier solution. The difference is considerable where the temperature gradient is large. This is to be expected.

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