Effect of Moving Boundaries on the Vibrating Elastic String

I-Shih Liu* & M. A. Rincon†

Instituto de Matemática, Universidade Federal do Rio de Janeiro

Caixa Postal 68530, Rio de Janeiro 21945-970, Brazil

Abstract

A new derivation of a wave equation for small vibrations of elastic strings fastened at ends varying with time is presented. The model takes into account the change of length during the vibration and the nonlinear behavior of elastic strings in general. This model is a generalization of the Kirchhoff equation which contains a nonlinear term involving the displacement gradient. Numerical simulations of the model are based on finite difference approximations. Differences between linear and nonlinear aspects and the assumptions of numerical and theoretical analysis are briefly discussed and comparisons are made for linear and nonlinear elastic strings as well as Kirchhoff model and the linear model without the term containing the displacement gradient.

Keywords: Kirchhoff equation, nonlinear elastic string, moving boundary, finite difference approximation, numerical simulation.

1 Introduction

One of the major classes of partial differential equations, the linear hyperbolic equation, describes, among other things, the transverse vibration of an elastic string. Let $u(x,t)$ be the transverse displacement of the string fastened at each end, $\alpha \leq x \leq \beta$, and assume that

1. The string is tightly stretched and the tension τ is uniform along the string.
2. The displacement gradient is small, $|\partial u/\partial x| \ll 1$, which ensures that the transverse displacement is small compared to the length of the string.
3. The string is uniform and made of a homogeneous material with constant mass density m.

Under these assumption, the mathematical model for small vibration of an elastic string proposed by D'Alembert is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c^2 = \frac{\tau}{m}, \tag{1}$$

in which by "tightly stretched" is understood that the tension τ is regarded as constant during the vibration irrespective of its small variation due to the transverse displacement of the string.

*e-mail: liu@dmm.im.ufrj.br

†e-mail: rincon@dcc.ufrj.br

A model which takes into account the small variation of tension due to the change of length of a string with fixed ends was later proposed in [1], known as the Kirchhoff equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{m}\left\{\tau_0 + \frac{\kappa}{2L_0}\int_\alpha^\beta \left(\frac{\partial u}{\partial x}\right)^2 dx\right\}\frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

where τ_0 is the tension and $L_0 = \beta - \alpha$ is the length of the string at rest; κ is the Young's modulus. The nonlinear term containing the displacement gradient has attracted a great deal of attention in mathematical analysis of the model by various authors, among them, Bernstein [2], Lions [3], Medeiros [4], Naraschinham [5], Strauss [6], Spagnolo [7].

In this paper we present a new derivation of the Kirchhoff model for nonlinear elastic strings fastened at ends varying with time more general than the one for linear elastic strings proposed by Medeiros *et al.* in [8]. We are mainly interested in the numerical aspects of the model. Comparisons are made for nonlinear and linear elastic strings as well as Kirchhoff model and the linear model without the term containing the displacement gradient.

2 Nonlinear Elastic String with Moving Ends

We shall maintain the basic assumptions listed above for small vibrations. In particular, the local variations of the tension during deformation of the string is neglected and only the average tension uniformly distributed throughout the string is considered. Similarly only the average strain is considered.

Let s be the length of the string, the average Lagrangean strain ε is then defined as

$$\varepsilon = \varepsilon(s) = \frac{s - L_0}{L_0}, \tag{3}$$

where L_0 is the length of the string at some reference state. Suppose that the constitutive relation between the tension and the strain of an elastic string is nonlinear in general given by

$$\tau = \sigma(\varepsilon). \tag{4}$$

It is assumed that the string is subjected to tension and the elastic modulus is nonnegative so that

$$\sigma(\varepsilon) > 0, \qquad \sigma'(\varepsilon) \geq 0. \tag{5}$$

We can write the average stress of the string at length s as

$$\tau = \tau(s) = \sigma(\varepsilon(s)). \tag{6}$$

Let the two ends of the string be fastened at $\alpha(t)$ and $\beta(t)$ on the left and the right respectively at the instant t. Suppose that the displacement of the string is in the vertical direction and is represented by $u(x,t)$, for $\alpha(t) \leq x \leq \beta(t)$.

The length of the string is given by

$$s(t) = \int_{\alpha(t)}^{\beta(t)} \left\{1 + \left(\frac{\partial u}{\partial x}\right)^2\right\}^{1/2} dx.$$

Since $|\partial u/\partial x| \ll 1$, by approximation we have

$$s(t) - \gamma(t) = \frac{1}{2}\int_{\alpha(t)}^{\beta(t)} \left(\frac{\partial u}{\partial x}\right)^2 dx, \tag{7}$$

where

$$\gamma(t) = \beta(t) - \alpha(t) \tag{8}$$

is the length of the horizontal projection of the string.

Considering small vibrations, by approximation we have

$$\tau(s(t)) - \tau(\gamma(t)) = \tau'(\gamma(t))\Big(s(t) - \gamma(t)\Big),$$

and by (7) we obtain the tension of the string

$$\tau = \tau(s(t)) = \sigma(\varepsilon(\gamma(t))) + \frac{1}{2L_0}\sigma'(\varepsilon(\gamma(t)))\int_{\alpha(t)}^{\beta(t)} \left(\frac{\partial u}{\partial x}\right)^2 dx.$$

Substituting into (1) we obtain

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{m}\left\{\sigma(\varepsilon(\gamma(t))) + \frac{1}{2L_0}\sigma'(\varepsilon(\gamma(t)))\int_{\alpha(t)}^{\beta(t)} \left(\frac{\partial u}{\partial x}\right)^2 dx\right\}\frac{\partial^2 u}{\partial x^2} = 0.$$

Therefore, we have obtained the following mathematical model for small vibration of an elastic string with moving ends:

$$\widehat{L}u(x,t) = \frac{\partial^2 u}{\partial t^2} - \left\{a(t) + b(t)\int_{\alpha(t)}^{\beta(t)} \left(\frac{\partial u}{\partial x}\right)^2 dx\right\}\frac{\partial^2 u}{\partial x^2} = 0, \tag{9}$$

for $\alpha(t) < x < \beta(t)$ and $t > 0$, where

$$a(t) = \frac{1}{m}\sigma(\varepsilon(\gamma(t))) > 0, \qquad b(t) = \frac{1}{2mL_0}\sigma'(\varepsilon(\gamma(t))) \geq 0, \tag{10}$$

since from (5) the stress and the elastic modulus are assumed to be positive.

When the two ends are fixed, the equation (9) reduces to the Kirchhoff equation (2) with $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ being constants, and

$$\tau_0 = \sigma(\varepsilon(\gamma)), \qquad \kappa = \sigma'(\varepsilon(\gamma)),$$

where κ is called the Young's modulus.

On the other hand, if the stress-strain relation is linear,

$$\tau = \sigma(\epsilon) = \tau_0 + \kappa\,\epsilon,$$

where κ is the Young's modulus and τ_0 is the stress at the reference state with length L_0, then from (10) we have

$$a(t) = \frac{\tau_0}{m} + \frac{\kappa}{m}\frac{\gamma(t) - L_0}{L_0}, \qquad b(t) = \frac{\kappa}{2mL_0}. \tag{11}$$

This corresponds to the model derived by Medeiros *et al.* in [8].[1]

[1] In [8] however, the strains are not correctly defined relative to the same reference state in the Hooke's law resulting in $b(t) = \dfrac{\kappa}{2m\gamma(t)}$.

3 Initial Value Problem with Moving Boundaries

Let $\widehat{Q} = \Big\{(x,t) \in I\!R^2;\ \alpha(t) < x < \beta(t),\ t > 0\Big\}$ denote the non-rectangular domain with boundary $\widehat{\Sigma} = \bigcup_{0<t<T} \{\alpha(t), \beta(t)\} \times \{t\}$. We consider the following problem:

$$
\textbf{(I)} \qquad \begin{cases} \widehat{L}u(x,t) = f(x,t) & \forall\,(x,t) \in \widehat{Q}, \\ u(x,t) = 0 & \forall\,(x,t) \in \widehat{\Sigma}, \\ u(x,0) = u_0(x), & \dfrac{\partial u}{\partial t}(x,0) = u_1(x) \qquad \alpha(0) \le x \le \beta(0), \end{cases}
$$

where the operator $\widehat{L}$ is defined in (9).

Mathematical analysis of the above problem has been considered in [8], in which the explicit forms of the functions $a(t)$ and $b(t)$ need not be specified. Therefore, their analysis remains valid for the more general model described in (9). The existence of local solutions is stated in the following theorem:

Theorem 3.1 *Let Ω_t e Ω_0 be the intervals $(\alpha(t), \beta(t))$ e $(\alpha(0), \beta(0))$ and consider the following hypotheses:*

i) $a \in W^{1,\infty}(0,\infty),\ \ a(t) \ge m_0 > 0.$

ii) $\alpha, \beta \in C^2([0,T]; I\!R)$;
$\alpha(t) < \beta(t);\ \ \alpha'(t) < 0,\ \ \beta'(t) > 0\ \ $ *for* $\ \ 0 \le t < T;$
$|\ \alpha'(t) + \gamma'(t)y\ | < (m_0/2)^{1/2}\ \ $ *for* $\ \ 0 \le t < T,\ \ 0 \le y \le 1.$

Then given $u_0 \in H_0^1(\Omega_0) \cap H^2(\Omega_0)$, $u_1 \in H_0^1(\Omega_0)$, $f \in C^0([0,T]; H_0^1(\Omega_t))$, there exists a unique solution of the problem (I), *$u : \widehat{Q} \to I\!R$, satisfying the following conditions for $0 < T_0 < T$:*

$u \in L^\infty(0, T_0; H_0^1(\Omega_t) \cap H^2(\Omega_t)),$

$u' \in L^\infty(0, T_0; H_0^1(\Omega_t)),$

$u'' \in L^2(0, T_0; L^2(\Omega_t)).$

Note that the function $b(t)$ must be non-negative for all t in the linear elastic model as a consequence of the positiveness of the Young's modulus. In the proof of the above theorem [8], the problem is transformed into an equivalent problem in a fixed rectangular domain, $Q = (0,1) \times (0,T)$, by the change of variables,

$$(x,t) \in \widehat{Q} \ \mapsto \ (y,t) \in Q, \qquad y = \frac{x - \alpha(t)}{\gamma(t)}. \tag{12}$$

Denoting $u(x,t) = v(y,t)$ and introducing the following operator defined on Q,

$$Lv(y,t) = \frac{\partial^2 v}{\partial t^2} + a(v,y,t)\frac{\partial^2 v}{\partial y^2} + b(y,t)\frac{\partial^2 v}{\partial t \partial y} + c(y,t)\frac{\partial v}{\partial y}, \tag{13}$$

we obtain the following equivalent problem in the rectangular domain $(0,1) \times (0,T)$:

$$
\textbf{(II)} \qquad \begin{cases} Lv(y,t) = g(y,t) & \forall\,(y,t) \in (0,1)\times(0,T), \\ v(0,t) = v(1,t) = 0, & 0 < t < T, \\ v(y,0) = v_0(y), & \dfrac{\partial v}{\partial t}(y,0) = v_1(y), \quad 0 \le y \le 1, \end{cases}
$$

where

$$
a(v,y,t) = \frac{1}{4}(b(y,t)^2 - \frac{1}{\gamma^2}\left\{a(t) + \frac{b(t)}{\gamma}\int_0^1 \left(\frac{\partial v}{\partial y}\right)^2 dy\right\}, \tag{14}
$$

$$
b(y,t) = -2\,\frac{\alpha' + \gamma' y}{\gamma}, \qquad c(y,t) = -\frac{1}{\gamma}\Big(\alpha'' + \gamma'' y + \gamma' b(y,t)\Big). \tag{15}
$$

For convenience, our numerical analysis using a finite difference approximation will be based on the equivalent problem in the rectangular domain.

4 Finite Difference Approximation

For numerical approximation, we shall rewrite the equation (13) in a more convenient form. Let

$$
u(y,t) = \frac{\partial v}{\partial t}(y,t), \qquad w(y,t) = \frac{\partial v}{\partial y}(y,t),
$$

then (13) is equivalent to the following system of the first order partial differential equations:

$$
\begin{cases} \dfrac{\partial u}{\partial t} + a(w,y,t)\dfrac{\partial w}{\partial y} + b(y,t)\dfrac{\partial u}{\partial y} = -c(y,t)w(y,t) + g(y,t), \\ \dfrac{\partial w}{\partial t} - \dfrac{\partial u}{\partial y} = 0. \end{cases} \tag{16}
$$

With matrix notations, $U = (u,w)^T$, the system takes the form,

$$
\frac{\partial U}{\partial t} + A(U,y,t)\frac{\partial U}{\partial y} = B(y,t)U + C(y,t), \tag{17}
$$

where

$$
A(U,y,t) = \begin{bmatrix} b(y,t) & a(w,y,t) \\ -1 & 0 \end{bmatrix}, \quad B(y,t) = \begin{bmatrix} 0 & -c(y,t) \\ 0 & 0 \end{bmatrix}, \quad C(y,t) = \begin{bmatrix} g(y,t) \\ 0 \end{bmatrix}. \tag{18}
$$

Let $U_j^n = U(jh, nk)$ be the approximate solution of the exact solution $U(y,t)$ of (17) at the discrete mesh points $y = jh$ and $t = nk$, where $0 \le j \le N$ and $n \ge 0$ are integers. We consider the approximate scheme of Lax-Friedrich [9],

$$
\frac{\partial U}{\partial t} \approx \frac{1}{k}\left(U_j^{n+1} - \frac{1}{2}(U_{j-1}^n + U_{j+1}^n)\right), \qquad \frac{\partial U}{\partial y} \approx \frac{1}{2h}\left(U_{j+1}^n - U_{j-1}^n\right).
$$

The system (17) becomes

$$
U_j^{n+1} = \frac{1}{2}\left(I - \lambda A_j^n\right)U_{j-1}^n + \frac{1}{2}\left(I + \lambda A_j^n\right)U_{j+1}^n + kB_j^n U_j^n + kC_j^n, \tag{19}
$$

for $n = 0, 1, 2, \cdots$; $j = 1, 2, \cdots, N-1$, where I denotes the identity matrix, $\lambda = k/h$ and

$$A_j^n = A(U_j^n, jh, nk), \qquad B_j^n = B(jh, nk), \qquad C_j^n = C(jh, nk).$$

In order to determine the approximate solution U_j^n from the above scheme (19) it is necessary to prescribe the initial values U_j^0 and the values of U_0^n e U_N^n at the boundary.

¿From (II) we have the initial condition,

$$U(y, 0) = (u(y, 0), w(y, 0))^T = (v_1(y), (v_0)_y(y))^T .$$

Since both functions $v_0(y)$ and $v_1(y)$ are prescribed for all $0 \leq y \leq 1$, the initial values of the approximate solutions U_j^0 are known for all $j = 0, \cdots, N$.

For the boundary values of the approximate solution, from (II) we have

$$U(0, t) = (u(0, t), w(0, t))^T = (0, v_y(0, t))^T , \quad U(1, t) = (u(1, t), w(1, t))^T = (0, v_y(1, t))^T ,$$

for which both $v_y(0, t)$ and $v_y(1, t)$ are not prescribed explicitly for $t > 0$. However using the identity,

$$\frac{\partial w}{\partial t}(y, t) = \frac{\partial u}{\partial y}(y, t),$$

for $t = nk$, $n > 0$, by approximation we have

$$v_y(0, t) \approx w_0^n = w_0^{n-1} + \lambda(u_1^{n-1} - u_0^{n-1}), \quad v_y(1, t) \approx w_N^n = w_N^{n-1} + \lambda(u_N^{n-1} - u_{N-1}^{n-1}),$$

for $n = 1, 2, \cdots$. Therefore the values of U_0^n and U_N^n can be determined from the approximate solution at the previous step by the use of the above relations.

To obtain the approximate solution of the original problem (I), we can first obtain the solution of the problem (II) by one of the following schemes:

$$v_j^n = v_{j-1}^n + \frac{h}{2}\left(w_j^n + w_{j-1}^n\right), \qquad v_j^n = v_{j+1}^n + \frac{h}{2}\left(w_j^n + w_{j+1}^n\right),$$

for any $n = 1, 2, \cdots$. Since $v_0^n = v_N^n = 0$ from the boundary condition, the values of v_j^n for $j = 1, \cdots, N-1$ can be obtained from either one of the above relations, or a combination of the two to get a better approximation. The approximate solution to the original problem (I) with moving ends can then be obtained by merely a change of variable.

4.1 Stability

We note that the presence of the non-linear term $A(U, y, t)$ in (17) makes the analysis of stability more difficult, however, since the deformation gradient $\partial u/\partial x$ is assumed to be small in the model, from the physical point of view, we would not expect any major difference between the linear and the nonlinear model. Therefore, in order to obtain some condition for stability, we shall neglect the non-linear term, by regarding $A(U, y, t) \approx A(y, t)$.

For the analysis of stability, the terms on the right hand side of the equation (17) are insignificant and thus will be omitted (see Theorem of perturbation of Strang [9]). Therefore, we consider the system

$$\frac{\partial U}{\partial t} + A(y, t)\frac{\partial U}{\partial y} = 0. \tag{20}$$

Since the coefficient $A(y,t)$ depends on y, we can not apply directly the convenient Fourier transform for the verification of the von Neumann condition of the numerical scheme. But we can use the following result for hyperbolic systems (see [9]): the initial value problem (20) is well-posed if and only if the problem with fixed coefficient,

$$\frac{\partial U}{\partial t} + A(\overline{y},t)\frac{\partial U}{\partial y} = 0 \tag{21}$$

is well-posed. Therefore, we shall analyze the stability of problem (21) with $\overline{y}$ fixed and write $A(t) = A(\overline{y},t)$. The numerical scheme similar to (19), is given by

$$U_j^{n+1} = \frac{1}{2}\left(I - \lambda A^n\right)U_{j-1}^n + \frac{1}{2}\left(I + \lambda A^n\right)U_{j+1}^n \tag{22}$$

for $n = 0, 1, \cdots$; $j = 1, \cdots, N-1$. The discrete Fourier transform of a function $U = \{U_j\}$ is given by

$$\widehat{U}(\xi) = \sum_j U_j e^{ij\xi}, \qquad i = \sqrt{-1}, \quad 0 \leq \xi < 2\pi.$$

Hence, we have

$$\{\widehat{U_{j-1}^n}\} = \sum_j U_{j-1}e^{ij\xi} = e^{i\xi}\widehat{U}^n, \qquad \{\widehat{U_{j+1}^n}\} = \sum_j U_{j+1}e^{ij\xi} = e^{-i\xi}\widehat{U}^n.$$

Applying the discrete transform to (22) we obtain

$$\begin{aligned}
\widehat{U}^{n+1}(\xi) &= \left(\frac{1}{2}\left(I + \lambda A^n\right)e^{i\xi} + \frac{1}{2}\left(I - \lambda A^n\right)e^{-i\xi}\right)\widehat{U}^n(\xi) \\
&= \left(\frac{1}{2}I(e^{i\xi} + e^{-i\xi}) + \frac{\lambda}{2}A^n(e^{i\xi} - e^{-i\xi})\right)\widehat{U}^n(\xi) \;=\; G(n,\xi)\widehat{U}^n(\xi),
\end{aligned}$$

where

$$G(n,\xi) = I\cos\xi - i\lambda A^n \sin\xi$$

is the amplification matrix.

Since the system is hyperbolic, for any fixed n there exists a real diagonal matrix $D^n = \operatorname{diag}(\mu_1^n, \mu_2^n)$ and a non-singular matrix Q^n such that $A^n = Q^n D^n (Q^n)^{-1}$. If we introduce

$$D(n,\xi) = I\cos\xi - i\lambda D^n \sin\xi,$$

then we have

$$G(n,\xi) = Q^n D(n,\xi)(Q^n)^{-1}.$$

To satisfy the condition of Von Neumann stability it is sufficient to show that

$$\|D(n,\xi)\| \leq 1. \tag{23}$$

Indeed, let $\mu^n = \max\{|\mu_1^n|, |\mu_2^n|\}$, then we have

$$\|D(n,\xi)\|^2 \leq |\cos\xi - i\lambda\mu^n \sin\xi|^2 = 1 - (1 - (\lambda\mu^n)^2)\sin^2\xi,$$

which implies the condition (23) when we take $\lambda = \lambda(n)$ satisfying the CFL (Courant-Friedrichs-Lewy) condition,

$$\lambda(n)|\mu^n| \leq 1.$$

Moreover, following the usual procedure, one can easily show that the numerical scheme is consistent with the system of equations of (II) with precision $O(k) + O(h)$.

4.2 Eigenvalues

We have seen that for stability $\lambda = k/h$ must satisfy the CFL condition, $|\mu_j|\lambda \leq 1$. From the numerical point of view it is important to determine the eigenvalues $\mu_j = \mu_j(y,t)$ of the matrix A defined in (17) in order to take a convenient value of λ. The characteristic equation of A is given by

$$\mu(y,t)^2 - b(y,t)\mu(y,t) + a(w,y,t) = 0,$$

which gives two different eigenvalues,

$$\mu_j(y,t) = \frac{1}{2}\left(b(y,t) \pm \sqrt{b(y,t)^2 - 4a(w,y,t)}\right), \quad j = 1,2.$$

Since the system is hyperbolic, the eigenvalues are real and hence it is necessary that $(b(y,t)^2 - 4a(w,y,t)) > 0$. By the use of (14), it follows that

$$b(y,t)^2 - 4a(w,y,t) = \frac{4}{\gamma(t)^2}\left(a(t) + \frac{1}{\gamma(t)}b(t)I_t\right) > 0, \tag{24}$$

where I_t stands for the integral,

$$I_t = \int_0^1 \left(\frac{\partial v}{\partial y}\right)^2 dy \geq 0.$$

As a consequence, the assumption (10) ensures the hyperbolicity. In particular, if the stress-strain relation is linear, from (11) the condition $a(t) > 0$ requires that

$$\gamma(t) > \gamma(0)\left(1 - \frac{\tau_0}{\kappa}\right), \tag{25}$$

which means that in order to maintain hyperbolicity for the vibration, the length of the string has a lower limit for any given initial tension. This is much weaker than the assumptions required by Theorem 3.1, in which not only $\gamma(t)$ must be a monotonically increasing function, but also $\alpha'(t) < 0$ and $\beta'(t) > 0$ are imposed. Such requirements seem unnecessary from the physical point of view. We shall consider a numerical example for a non-monotone function $\gamma(t)$ later.

With (14) and (15) for $a(y,t)$ and $b(y,t)$, the eigenvalues μ_j become

$$\mu_j(y,t) = \frac{1}{\gamma(t)}\left((\alpha'(t) + \gamma'(t)y) \pm \sqrt{a(t) + \frac{1}{\gamma(t)}b(t)I_t}\right), \quad j = 1,2.$$

Let

$$\mu(t) = \max_{\substack{y\in[0,1]\\ j=1,2}} |\mu_j(y,t)|. \tag{26}$$

¿From the CFL condition for every time-step k, we can take $\lambda = k/h \leq |\ \mu\ |^{-1}$. Consequently, we can adopt a non-uniform time-step k with variable $\lambda(t)$. In this case when $\mu(t)$ is very small, the step k will be very large resulting in a great loss of precision. Therefore, in the numerical scheme we can set an upper bound for the value of λ to maintain a reasonable precision. If a uniform time-step is preferred, it is sufficient to set $\lambda = \min_{t\in(0,T)} \lambda(t)$. The integral I_t can be calculated with a convenient numerical method from the approximate solution at the previous step.

5 Numerical Simulations

Several numerical examples will be shown here to illustrate some features of the present model. In these examples, we set both the external force and the initial velocity to be zero, namely, $f(x,t) = 0$ and $u_1(x) = 0$ in (I), or equivalently $v_1(y) = 0$ in (II).

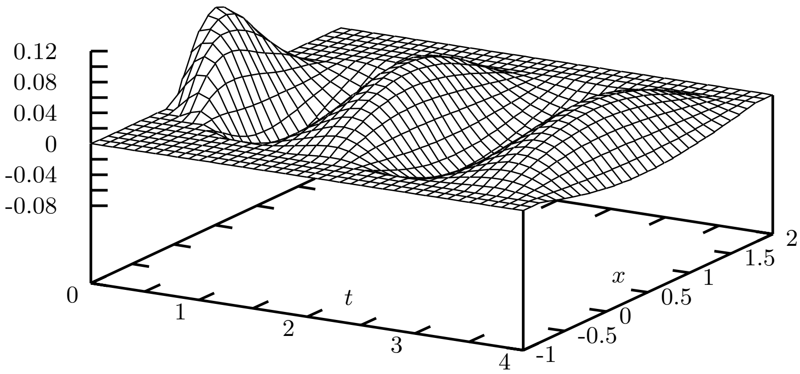

Figure 1: Example 1 – Vibration $u(x,t)$ with moving ends

5.1 Example 1: Linear string

We first consider the linear stress-strain relation (11) with the following data: $\tau_0/m = 1$, $\kappa/m = 5$, $h = 0.005$ and a fixed $\lambda = 0.2$. The initial position of the string is taken to be the smooth function given by

$$u(x,0) = \frac{1}{\pi^2} \sin \pi x.$$

The string is pulled at both ends satisfying the assumptions (ii) of Theorem 3.1,

$$\alpha(t) = -\frac{t}{t+1}, \qquad \beta(t) = \frac{2t+1}{t+1},$$

Note that $\lim_{t\to\infty} \alpha(t) = -1$, $\lim_{t\to\infty} \beta(t) = 2$, that is, both ends tend to be fixed as time increases. Hence the maximum eigenvalue is bounded for all t and the numerical scheme converges for any T.

In Fig. 1 the evolution of the function $u(x,t)$, the position of the string at subsequent times, is plotted, showing the profile of the string and the moving ends.

5.2 Example 2: Non-linear string

We now consider a nonlinear stress-strain relation of the string given by

$$\sigma(\varepsilon) = \kappa(\varepsilon + \varepsilon_0)^{1/2}$$

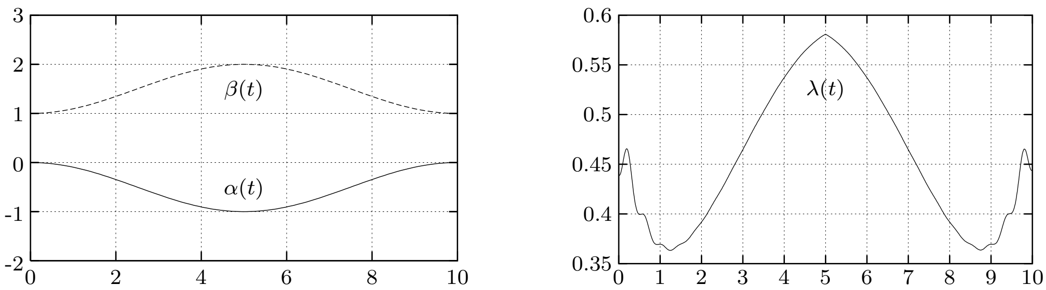

Figure 2: Example 2 – Moving ends $\alpha(t)$, $\beta(t)$ and variable time-step

with the data chosen for convenience only (for some experimental data see [11]),

$$\kappa = 8\sqrt{5}, \qquad \varepsilon_0 = 1/20.$$

The initial condition is given by

$$u(x, 0) = \frac{1}{10} \sin \pi x,$$

and the moving ends are specified by the functions,

$$\alpha(t) = \frac{1}{2}\left(\cos\frac{2\pi t}{T} - 1\right), \qquad \beta(t) = \frac{1}{2}\left(3 - \cos\frac{2\pi t}{T}\right),$$

which are not monotone functions as required by Theorem 3.1. The string has the same length, $\gamma(0) = 1$ and $\gamma(T) = 1$, at the initial and the final time $T = 10$, as shown in Fig. 2. The numerical scheme does not require the monotonicity of those functions.

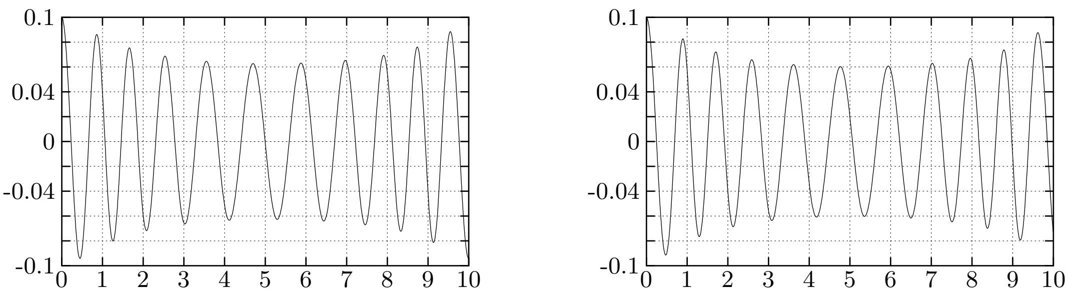

Figure 3: Example 2: $u(0.5, t)$ for Kirchhoff and linear models

Variable time-steps are used in this simulation, $\lambda(t) = \mu(t)^{-1}$ with $h = 0.005$. The variation of λ plotted against t is also shown in Fig. 2. The solution for the string, $u(x, t)$, for $x = 0.5$ and $0 \le t \le T$ is plotted on the left side of Fig. 3. Note that the amplitude becomes smaller as the length of the string increases and vice versa.

In order to see the influence of the nonlinear term containing the displacement gradient in the Kirchhoff equation, the solution without this term is plotted on the right side of Fig. 3 for comparison. No essential difference between the nonlinear and linear model can be observed in this example.

5.3 Example 3: Strings with fixed ends

The objective of this example is to show the influence of the nonlinear term in the Kirchhoff model with fixed ends. The non-linear stress-strain relation and the data are the same as in Example 2, except of course, that $\alpha(t) = 0$ and $\beta(t) = 1$.

Fig. 4 shows both the vibration of the string at $x = 0.5$ and the variation of the eigenvalue $\mu(t)$ given by

$$\mu(t) = \left\{ a(t) + b(t) \int_{\alpha(t)}^{\beta(t)} \left(\frac{\partial u}{\partial x} \right)^2 dx \right\}^{1/2} .$$

The amplitude of the vibration is constant as expected, however unlike the linear model, the characteristic speed of vibration is not constant, it changes periodically between the values at the peak of the amplitude.

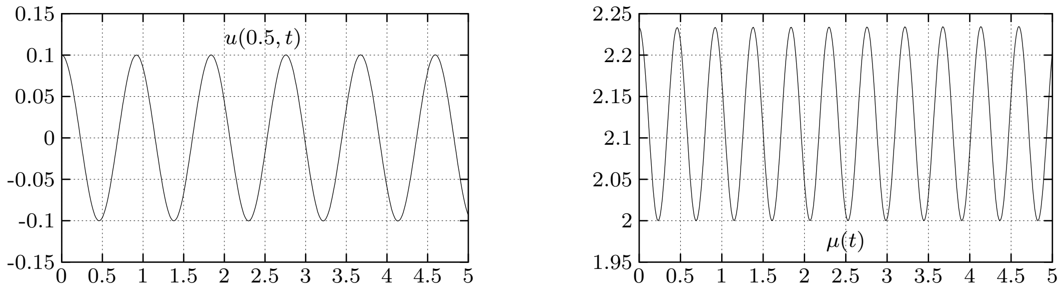

Figure 4: Example 3 – Displacement $u(0.5, t)$ and eigenvalue

5.4 Example 4: Regularity of initial data

We consider the initial condition given by a function, which is only continuous,

$$u(x, 0) = \begin{cases} \dfrac{1}{10} x^2 & 0 \le x \le 0.5 \\ \dfrac{1}{10} (x-1)^2 & 0.5 \le x \le 1. \end{cases} \tag{27}$$

Obviously, $u_0(x) \in H_0^1(0,1)$ but $u_0(x) \notin H^2(0,1)$, and hence it does not satisfy the assumption of Theorem 3.1. However, from the numerical simulations, it seems that the initial data need not be so smooth, as it is shown in this example.

We take boundary functions, satisfying the assumption (ii) of the theorem, given by

$$\alpha(t) = -t, \qquad \beta(t) = t + 1$$

and consider the linear model as in Example 1 with $\tau_0/m = 1$, $\kappa/m = 5$, $h = 0.01$ and a fixed $\lambda = 0.5$.

The numerical solution $u(x,t)$ is determined for the interval $[0,T] = [0,25]$. Fig. 5 shows the position of the string at $t = 0$, 0.25, 0.50, 0.75, 1.00 during which the length of string has varied from 1 to 3. The variations of the string at $x = 0.5$ and the maximum eigenvalue are shown in Fig. 6 for the time interval $0 \le t \le 25$.

Note that the non-regularity of the solution at the beginning, due to the initial condition which is only continuous, tends to be smoothed out in time. The decreasing of the maximum

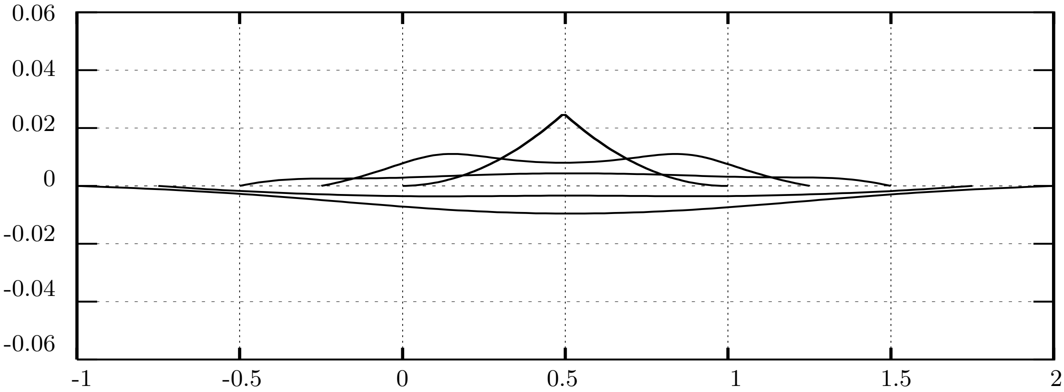

Figure 5: Example 4 – $u(x,t)$ for $t = 0$, 0.25, 0.50, 0.75, 1.00

eigenvalue $\mu(t)$ may seem strange as the string is pulled. But this is by no means a typical situation. Moreover, $\mu(t)$ calculated from (26) is not the same as the eigenvalue of problem (I), and in this example the change of length is enormous. Indeed, the data used are unrealistic because the length at time $t = 25$ is more than 50 times the original length. Our purpose is just to show some numerical aspects of the mathematical model.

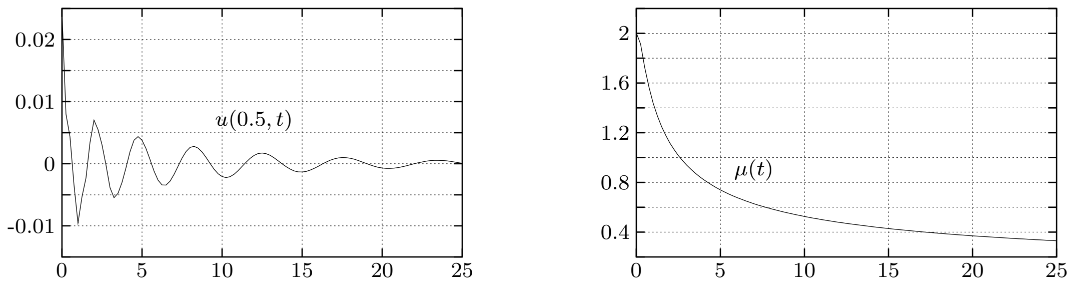

Figure 6: Example 4 – Displacement at midpoint and maximum eigenvalue

6 Final Remarks

We have derived a Kirchhoff model with moving ends more general than the one proposed in [8] for linear elastic strings. The results for nonlinear elastic string in general differs slightly from those of the linear one and they are qualitatively similar if small vibrations are concerned.

The nonlinear term containing the displacement gradient, $\int |\partial u/\partial x|^2 dx$, is the essential mathematical feature of the Kirchhoff model. However, since $|\partial u/\partial x|$ is assumed to be small in the model, from the physical point of view, we would not expect any major difference between the linear and the nonlinear model as illustrated in Fig. 3. Nevertheless, the nonlinear term gives rise to new challenges in mathematical analysis. Most of the theoretical results rely on assumptions, such as the monotonicity of the non-cylindrical domain and smoothness of the initial data required in Theorem 3.1. Such assumptions are much stronger than the requirements for the numerical simulations as we have seen in the examples presented in this paper. Those are still open questions in analysis for this model.

Acknowledgments: *The author (ISL) acknowledges the partial support for research from CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico) of Brazil.*

References

[1] G. Kirchhoff, Vorlesungen über Mechanik, Tauber, Leipzig, 1883.

[2] S.Bernstein, Sur une classe d'équations fonctionelles aux derivées partielles, Isv. SSSR, Série Math. 4 (1940), 17-26.

[3] J.L. Lions, Quelques métodes de résolution des problèmes aux limites non-linéaires, Dunod, Paris, 1969.

[4] L.A. Medeiros, On some nonlinear pertubation of Kirchhoff-Carrier operator, Comp. and Appl. Math 13 (1994), 225-233.

[5] R. Narashinham, Nonlinear vibrations of an elastic string, J. Sound. Vibr. 8 (1968), 134-136.

[6] W.A. Strauss, Nonlinear wave equations, George Mason University, USA, 1989.

[7] S. Spagnolo, The Cauchy problem for Kirchoff equations. Rendioconti del Seminario Matematico e Fisico di Milano 62 (1992), 17-51.

[8] L.A. Medeiros and J. Limaco F., Kirchhoff-Carrier Elastic Strings in noncylindrical Domains. Portugaliae Mathematica 56 (1999), 1-36.

[9] G.A. Sod, Numerical Methods in Fluid Dynamicas (Initial and Initial Boundary-Value Problems), Cambridge University Press, 1985.

[10] W.G. Newman, Global solution of a nonlinear string equation, J. Math. Anal. Appl. 192 (1995), 689-704.

[11] J.F. Bell, The experimental foundation of solid mechanics, *Mechanics of Solids I*, Ed. by C. Truesdell, Handbuch de Physik, Vol. VIa/1, Springer, 1973.