NONHOLONOMIC SIMPLE $\mathcal{D}$-MODULES OVER PROJECTIVE VARIETIES

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Abstract. Let X be a smooth complex projective variety whose Picard group is isomorphic to $\mathbb{Z}$. We prove that there exist simple nonholonomic $\mathcal{D}_X$-modules of holonomic defect 1.

Let X be an irreducible smooth algebraic variety of dimension n over $\mathbb{C}$ and let $\mathcal{D}_X$ be the sheaf of rings of differential operators over X. A coherent sheaf of $\mathcal{D}_X$-modules is known simply as a $\mathcal{D}_X$-module. The most important invariant of a $\mathcal{D}_X$-module $\mathcal{M}$ is its characteristic variety $\mathrm{Ch}(\mathcal{M})$, which is a co-isotropic subvariety of the cotagent bundle T^*X. In particular,

$$\dim \mathrm{Ch}(\mathcal{M}) - n \geq 0.$$

This difference is known as the *holonomic defect* of $\mathcal{M}$. A $\mathcal{D}_X$-module is *holonomic* if its defect is zero, and *nonholonomic* otherwise. See [2] for more details.

Simple holonomic modules are easy to produce on an industrial scale over any given X. By contrast very few examples of nonholonomic simple $\mathcal{D}_X$-modules are known when X is an affine variety, and up-till-now none whatsoever were known when X was projective. The only exception to this rule is the affine space, for which J. Bernstein and V. Lunts [1] developed a very successful geometrical approach to constructing nonholonomic simple modules of holonomic defect $n-1$. By recasting the problem in terms of properties of the characteristic variety, they established a bridge between the theory of $\mathcal{D}$-modules and the theory of holomorphic foliations.

In this note we use this bridge to prove that there exist nonholonomic $\mathcal{D}$-modules over projective varieties. Throughout the note $\mathrm{Pic}(X) = \mathbb{Z}$ means that the Picard group of the complex projective algebraic variety X is the free abelian group generated by the class of the hyperplane section. The main theorem is the following.

Theorem 1. *There exist nonholonomic $\mathcal{D}_X$-modules of holonomic defect* 1 *over every smooth complex projective variety X with* $\mathrm{Pic}(X) = \mathbb{Z}$ *and* $\dim(X) \geq 2$.

Throughout the note we assume that the n-dimensional projective space $\mathbb{P}^n$ has homogeneous coordinates $x_0, \ldots, x_n$, and that U_j is the open set of $\mathbb{P}^n$ defined by $x_j \neq 0$. Moreover, we always identify U_j with $\mathbb{C}^n$ by taking $x_j = 1$.

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If $\mathcal{F}$ is a foliation of dimension one of a smooth complex algebraic variety X, then $\mathrm{Sing}(\mathcal{F})$ denotes its set of singular points. Let $p \in X$. Then there exists an affine neighbourhood U_p of p such that $\mathcal{F}$ is defined on U_p by a derivation d_p of $\mathcal{O}(U_p)$, the coordinate ring of U_p. If p is a singularity of $\mathcal{F}$, set

$$\Lambda_p = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2,$$

where λ_1 and λ_2 are the eigenvalues of the 1-jet of d_p at p.

One of the key ingredients of the proof is the following generalization of a result of J. Bernstein and V. Lunts [1, section 4]. For a proof see [4, Theorem 3.5].

Theorem 2. *Let W be a smooth affine algebraic surface over $\mathbb{C}$ whose Picard group is trivial. Suppose that d is a derivation of W with no invariant algebraic curves and a finite nonempty singularity set. If $f \in \mathcal{O}(W)$ is such that $f(p) \notin \Lambda_p$ for all $p \in \mathrm{Sing}(d)$, then $\mathcal{D}(W)/\mathcal{D}(W)(d+f)$ is a nonholonomic simple $\mathcal{D}(W)$-module of dimension 3.*

We also use the following lemma, which allows us to lift the result from surfaces to varieties of higher dimension.

Lemma 3. *Let X be a smooth projective variety with $\mathrm{Pic}(X) = \mathbb{Z}$. There exists a smooth surface $S \subseteq X$ with $\mathrm{Pic}(S) = \mathbb{Z}$.*

Proof. Let $X \subseteq \mathbb{P}^n$ and let H be a generic hypersurface of $\mathbb{P}^n$ of degree $k > 0$. It follows from Bertini's Theorem that $H \cap X$ is a smooth ireducible projective variety of dimension $\dim(X) - 1$.

We must now show that $\mathrm{Pic}(X \cap H) = \mathbb{Z}$. If $\dim(X) > 3$ and $k = 1$ this is a consequence of [9, Theorem 6.5, p. 246]. If $\dim(X) = 3$, we must choose $k \gg 0$ and apply [9, Theorem 7.5, p. 247]. □

Proof of Theorem 1. We will assume, throughout the proof, that $X \subseteq \mathbb{P}^n$, where $n \geq 2$, and that $\mathcal{I}$ is the ideal of X in $\mathcal{O}_{\mathbb{P}^n}$.

Let $k > 0$ be an integer and denote by Θ_X the tangent sheaf of X. A section θ of $\Theta_X(k)$, induces a one dimensional foliation $\mathcal{F}$ on X. Suppose that θ and k have been chosen such that:

(1) $\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$ maps onto $\mathrm{H}^0(X, \mathcal{O}_X(k))$,
(2) $\mathcal{F}$ does not have any invariant algebraic curves, and
(3) the singular set of $\mathcal{F}$ is necessarily nonempty.

If $k \gg 0$ and if θ is a generic section of $\Theta_X(k)$, then (1) follows from Serre's Theorem [7, Theorem 5.2, p. 228] and (2) and (3) from [6, Theorem 1.1] and [6, Theorem 4.3], respectively. Moreover, without loss of generality, we may assume that $\mathrm{Sing}(\mathcal{F}) \cap U_j \neq \emptyset$ for each $0 \leq j \leq n$.

Let $q \in X$ and $r \in \mathbb{C}$, then

$$V_{j,r,q} = \{G \in \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) : (G - r x_j^k)(q) \neq 0\}$$

is an open dense subset of the affine space $\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$. The same is true of the set W formed by those $G \in \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$ which are irreducible homogeneous polynomials. Moreover, the set Y of triples (j, q, r) where $q \in \mathrm{Sing}(\mathcal{F})$, $r \in \Lambda_q$ and $0 \leq j \leq n$ is countable. Therefore,

$$\mathcal{U} = W \cap \Big(\bigcap_{(j,r,q) \in Y} V_{j,r,q} \Big),$$

is dense in $\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$. Let

$$G_{|U_j} = G(x_1, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_n).$$

Note that, if $G \in \mathfrak{U}$ and $q \in U_j \cap \mathrm{Sing}(\mathcal{F})$, then $G_{|U_j}(q) \notin \Lambda_q$. In particular, $G(q) \neq 0$. Since $\mathrm{H}^0(X, \mathcal{I}(k))$ generates $\mathcal{I}$ for $k \gg 0$, this implies that the image of G in $\mathrm{H}^0(X, \mathcal{O}_X(k))$ cannot be zero.

Let F be the image in $\mathrm{H}^0(X, \mathcal{O}_X(k))$ of an element of $\mathfrak{U}$. The section $\theta + F \in \mathrm{H}^0(X, \Theta_X(k) \oplus \mathcal{O}_X(k))$ gives rise to a map

$$\phi : \mathcal{O}_X(-k) \to \Theta_X \oplus \mathcal{O}_X \subset \mathcal{D}_X.$$

Let $\mathcal{E}$ be the image of ϕ in $\mathcal{D}_X$, and put

$$\mathcal{M} = \mathcal{D}_X / \mathcal{D}_X \mathcal{E}.$$

Since ϕ is an injective map of $\mathcal{O}_X$-modules, it follows that $\mathcal{E} \cong \mathcal{O}_X(-k)$. In particular, $\Gamma(U_j \cap X, \mathcal{E})$ is a rank one free $\mathcal{O}(U_j \cap X)$-module generated by $d_j + f_j$, the dehomogeneization of $\theta + F$ with respect to x_j, for every $0 \leq j \leq n$. Moreover, the hypotheses on θ and F imply that

(i) d_j does not have invariant algebraic curves;
(ii) d_j has a singularity in $U_j \cap X$; and
(iii) $f_j(p) \notin \Lambda_p$ for all $p \in \mathrm{Sing}(d_j) \cap U_j$.

Note also that $\mathrm{Pic}(X) = \mathbb{Z}$ implies that $\mathrm{Pic}(X \cap U_j)$ is zero. Thus, assuming that $\dim(X) = 2$, we have by Theorem 2 that

$$M_j = \Gamma(U_j \cap X, \mathcal{M}) = \mathcal{D}(U_j \cap X)/\mathcal{D}(U_j \cap X)(d_j + f_j)$$

is a simple nonholonomic module of dimension 3.

We must show that $\mathcal{M}$ itself is a simple $\mathcal{D}_X$-module. Suppose that $\mathcal{Q}$ is a nonzero quotient of $\mathcal{M}$. Let Y be the support of $\mathcal{Q}$ in X. Then, for some $0 \leq j \leq n$, we have that $Y \cap U_j \neq \emptyset$ and that $\mathcal{Q}_{|U_j \cap X} \neq 0$. Since $\mathcal{M}_{U_j \cap X}$ is a simple $\mathcal{D}_{|U_j \cap X}$-module, it follows that $\mathcal{Q}_{|U_j \cap X} = \mathcal{M}_{|U_j \cap X}$. But the support of $\mathcal{M}$ is equal to X, so that Y contains $U_j \cap X$. Taking into account that Y is a closed subset of X, we deduce that $Y = X$. Thus, if $S = \{j : U_j \cap X \neq \emptyset\}$, then $\mathcal{Q}_{|U_j \cap X} \neq 0$ for all $j \in S$. Hence, $\mathcal{Q}_{|U_j \cap X} = \mathcal{M}_{|U_j \cap X}$, for all $j \in S$. Therefore, $\mathcal{Q} = \mathcal{M}$, and Theorem 1 is proved when $\dim(X) = 2$.

Suppose now that $\dim(X) > 2$. It follows from Lemma 3 that X contains a smooth surface S with $\mathrm{Pic}(S) = \mathbb{Z}$. But we have already seen that there exists a simple $\mathcal{D}_S$-module $\mathcal{M}$ of dimension 3 (hence holonomic defect 1). If $i : S \to X$ is the canonical embedding then, by Kashiwara's Equivalence, the direct image $i_+(\mathcal{M})$ is a simple $\mathcal{D}_X$-module with holonomic defect 1, [2, Theorem 7.11, p.263]. In particular this module is nonholonomic, and the proof is complete.

In principle, Theorem 1 can be used to construct explicit examples of simple nonholonomic $\mathcal{D}_X$-modules. However, in order to do that, one must have access to a foliation of X without algebraic solutions. Although a generic foliation will have this property, explicit examples are known only over a few simple projective varieties. Luckily, the projective plane is one of these. Taking into account the fact that a $\mathcal{D}_{\mathbb{P}^n}$-module is generated by its global sections, we give an explicit example of a module over the ring $\Gamma(\mathbb{P}^n, \mathcal{D}_{\mathbb{P}^n})$ which localizes to a nonholomic simple $\mathcal{D}_{\mathbb{P}^n}$-module.

Example. Let $m \geq 3$. Consider a map

$$\phi : \mathcal{O}(1-m) \to \Theta_{\mathbb{P}^2} \oplus \mathcal{O},$$

and denote its image by $\mathcal{I}$. Of course, $\mathcal{I} \cong \mathcal{O}(1-m)$, as $\mathcal{O}_{\mathbb{P}^2}$-modules. Let U_j be as in the proof of Theorem 1. Then, $\Gamma(U_j, \mathcal{I})$ is generated over $\mathcal{O}(U_j)$ by $d_j + f_j$, where d_j is a derivation and $f_j \in \mathcal{O}(U_j)$. If $\mathcal{O}(U_0) = \mathbb{C}[t_1, t_2]$, where $t_1 = x_1/x_0$ and $t_2 = x_2/x_0$, then,

$$\mathcal{O}(U_j) = \mathbb{C}[t_1 t_j^{-1}, t_2 t_j^{-1}, t_j^{-1}] \text{ for } j = 1 \text{ and } 2,$$

as subrings of the quotient ring of $\mathcal{O}(U_0)$. Writing d_j in the coordinates of $\mathcal{O}(U_0)$ we have that $d_j = t^{1-m}(d_0 + f_0)$. Therefore,

$$\Gamma(U_0, \mathcal{I}) = \mathcal{O}(U_0)(d_0 + f_0) \text{ and } \Gamma(U_j, \mathcal{I}) = \mathcal{O}(U_j) t_j^{1-m}(d_0 + f_0),$$

for $j = 1, 2$. Hence, $\Gamma(\mathbb{P}^2, \mathcal{D}_{\mathbb{P}^2}\mathcal{I}) = \mathcal{D}(1-m)(d_0 + f_0)$ where

$$\mathcal{D}(1-m) = \mathcal{D}(U_0) \cap \mathcal{D}(U_1) t_1^{1-m} \cap \mathcal{D}(U_2) t_2^{1-m}.$$

We recall the explicit construction of $\mathcal{D}(1-m)$ given in [5]. Let A_2 be the Weyl algebra generated over $\mathbb{C}$ by t_1, t_2, $\partial_1 = \partial/\partial t_1$ and $\partial_2 = \partial/\partial t_2$. If $\pi = t_1\partial_1 + t_2\partial_2$, set

$$M(k) = \sum_{i=1}^{2} \mathcal{D}_k \partial_i + \mathcal{D}_k(\pi - k - 1) = \sum_{i=1}^{2} \partial_i \mathcal{D}_{k+1} + (\pi - k - 1)\mathcal{D}_{k+1},$$

where $\mathcal{D}_k$ is the $\mathbb{C}$-subalgebra of A_2 generated by ∂_j, $t_i\partial_j$ and $t_i(\pi - k)$, for $1 \leq i, j \leq 2$. Then, by [5, Theorem 4.7, p. 216],

$$\mathcal{D}(1-m) = \prod_{j=0}^{m-2} M(j).$$

Note that $\mathcal{D}_k$ is a twisted ring of differential operators over $\mathbb{P}^2$.

In order to carry out the construction, we need now to choose a foliation without invariant algebraic curves. The simplest foliation of this type over $\mathbb{P}^2$ is the one defined by J. P. Jouanolou in [8, p. 157ff]. This foliation is generated over U_0 by

$$d_0 = t_2^m \pi - \partial_1 - t_1^m \partial_2.$$

All the singularities of the Jouanolou foliation belong to $U_0 \cap U_1 \cap U_2$. A simple computation shows that the t_2-coordinate of these singularities are the roots of $t_2^{m^2+m+1} - 1 = 0$. Moreover, the eigenvalues of the 1-jets of d_0 at each one of its singularities are

$$\lambda = \frac{-(m+2) + \imath m\sqrt{3}}{2},$$

and its conjugate. Therefore, $\Lambda_p = \mathbb{Z}\lambda + \mathbb{Z}\overline{\lambda}$, for all $p \in \mathrm{Sing}(d_0)$.

Now, let L be the extension of $\mathbb{Q}$ obtained by adjoining λ to the splitting field of $t_2^{m^2+m+1} - 1$. Choose $\xi \in \mathbb{C} \setminus L$, and let

$$F(x, y, z) = \xi(x^{m-1} + y^{m-1} + z^{m-1}).$$

Thus, in the notation of the proof of Theorem 1, we have that $f_j(p) \notin L$ for all $p \in \mathrm{Sing}(d_0)$. Since $\Lambda_p \subset L$, it follows that $f_j(p) \notin \Lambda_p$ for all $p \in \mathrm{Sing}(d_0)$. Therefore, f_j satisfies condition (iii) of Theorem 1 for $0 \leq j \leq 2$.

Here is a simple strategy that can be used to choose ξ. Let p and q be positive prime numbers, and assume that $p > 2(m^2 + m)$. Choose ξ to be a p-th root of q.

Since $[L:\mathbb{Q}] \leq (2(m^2+m))!$ it follows, by the degree formula for field extensions, that $\xi \notin L$, as required.

The exactness of the global sections functor in the category of $\mathcal{D}_{\mathbb{P}^2}$-modules [3, Theorem 1.9, p. 14], allows us to write the module constructed in Theorem 1 in the form

$$M = \mathcal{D}(\mathbb{P}^2)/\Gamma(\mathbb{P}^2, \mathcal{D}_{\mathbb{P}^2}\mathcal{I}).$$

But we have already seen that

$$\Gamma(\mathbb{P}^2, \mathcal{D}_{\mathbb{P}^2}\mathcal{I}) = \prod_{j=0}^{m-2} M(j)(d_0 + f_0).$$

Thus, $D_{\mathbb{P}^2} \otimes_{D(\mathbb{P}^2)} M$ is an explicit example of a simple nonholonomic $D_{\mathbb{P}^2}$-module.

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