London Mathematical Society ISSN 1461–1570

ON HOMOGENOUS MINIMAL INVOLUTIVE VARIETIES

L. C. O. ALMEIDA AND S. C. COUTINHO

Abstract

Let $\mathcal{S}(2n, k)$ be the variety of homogeneous polynomials of degree k in $2n$ variables. The authors of this paper give a computer-assisted proof that there is an analytic open set Ω of $\mathcal{S}(4, 3)$ such that the surface $F = 0$ is a minimal homogeneous involutive variety of $\mathbb{C}^4$ for all $F \in \Omega$. As part of the proof, they give an explicit example of a polynomial with rational coefficients that belongs to Ω.

1. *Introduction*

The study of the homogeneous involutive varieties of $\mathbb{C}^{2n}$ began in 1988 with the work of J. Bernstein and V. Lunts [**3**]. Their interest in these varieties was prompted by the fact that they appear naturally as characteristic varieties of modules over the Weyl algebra. This is the (noncommutative) complex algebra A_n generated by the coordinate functions $x_1, \ldots, x_n$ and the differential operators $\partial/\partial x_1, \ldots, \partial/\partial x_n$.

The word 'involutive' here refers to the behaviour of these varieties with respect to the standard symplectic structure of $\mathbb{C}^{2n}$ given by the 2-form $\omega = \sum_{i=1}^{n} dx_{i+n} \wedge dx_i$. This form defines a Poisson bracket in the polynomial ring $\mathbb{C}[x_1, \ldots, x_{2n}]$. First, to a polynomial $f \in \mathbb{C}[x_1, \ldots, x_{2n}]$, we associate the *hamiltonian vector field* h_f by the formula $\omega(\cdot, h_f) = df$. The *Poisson bracket* is now defined by $\{f, g\} = \omega(h_f, h_g)$. An algebraic variety $X \subseteq \mathbb{C}^{2n}$ is *involutive* if its ideal $I(X)$ is closed under the Poisson bracket; that is, if $\{I(X), I(X)\} \subseteq I(X)$. See [**5**, Chapter 1] for more details.

A celebrated theorem in the theory of $\mathcal{D}$-modules states that the characteristic variety of a finitely generated A_n-module is always involutive. Moreover, if we endow A_n with the filtration obtained by giving degree 1 to both the x_i and the $\partial/\partial x_i$, then the characteristic variety of an A_n-module computed with respect to this filtration will be a homogeneous subvariety of $\mathbb{C}^{2n}$, in the sense that its ideal is homogeneous with respect to the usual grading of the polynomial ring.

In their work in [**3**], Bernstein and Lunts were led to consider homogeneous involutive varieties of $\mathbb{C}^{2n}$ that are *minimal* in the sense that they do not contain a proper homogeneous involutive subvariety. They showed that (apart from an extra, mild hypothesis) if a finitely generated A_n-module has such a minimal homogeneous involutive variety for its characteristic variety, then it must be simple.

Since an involutive variety must have dimension greater than or equal to n, all irreducible homogeneous involutive varieties of dimension n must be minimal. The main result of [**3**] is the following theorem.

The research of the first author was supported by a PIBIC/CNPq Scholarship. The research of the second author was partially supported by grants from CNPq and PRONEX (Commutative algebra and algebraic geometry).
Received 8 December 2003, revised 18 July 2005; *published* 23 December 2005.
2000 Mathematics Subject Classification 37F75, 34M45 (primary), 53D05, 13P10 (secondary).

THEOREM 1.1. *If $F \in \mathbb{C}[x_1, x_2, x_3, x_4]$ is a homogeneous generic polynomial of degree $k \geqslant 4$, then the hypersurface $\mathcal{Z}(F)$ is a minimal involutive homogeneous variety of $\mathbb{C}^4$.*

We must explain what we mean by 'a generic polynomial' in this context. First of all, we may identify the space $\mathcal{S}(2n, k)$ of all homogeneous polynomials in $2n$ variables and degree k with the affine space of dimension $\binom{2n+k}{k}$. Then, 'general' means that the set of polynomials F for which $\mathcal{Z}(F)$ is *not* a minimal involutive homogeneous variety is contained in a countable union of hypersurfaces of $\mathcal{S}(2n, k)$.

This result was later generalized by Lunts [15] to all $n \geqslant 2$ and $k \geqslant 4$, and by T. McCune [16] to $k = 3$ and $n = 2$. It should be pointed out that although these results imply that 'most' polynomials of degree $k \geqslant 3$ give rise to minimal involutive hypersurfaces in $\mathbb{C}^4$, the proofs given in [3], [15] and [16] do not allow one to write down any explicit examples of such polynomials — say, one with rational coefficients, with which one might try a few computations.

In order to prove Theorem 1.1, Bernstein and Lunts look at the direction field induced on the complex projective space $\mathbb{P}^3$ by the hamiltonian vector field h_F of $\mathbb{C}^4$. This places the problem in the framework of the theory of holomorphic foliations, and allows one to use all the machinery that has been developed in this field. Indeed, similar problems have been studied for foliations over projective space for many years, notably by Jouanolou, Lins Neto and Soares [11, 13, 14, 18].

Our aim in this paper is twofold. First, we use methods from symbolic and numerical computation to obtain an example of a polynomial of degree 3 such that $\mathcal{Z}(F)$ is a minimal involutive homogeneous subvariety of $\mathbb{C}^4$. Then we use this example to prove the following theorem.

THEOREM 1.2. *There exists an open analytic dense subset Ω of $\mathbb{P}(\mathcal{S}(4, 3))$ such that the hypersurface $\mathcal{Z}(F)$ is minimal involutive homogeneous for every $F \in \Omega$.*

We prove the theorem using the method developed by Lins Neto in [13], together with an index theorem for singular foliations on surfaces proved by Suwa [19, Theorem 2.1]. The same strategy can also be applied to polynomials of degree 4, the only constraint being the time taken by the computations. However, in order to apply it to polynomials of degree higher than 4, one would have to generalise Proposition 3.7.

The paper is divided into six sections. Section 2 contains a summary of some of the basic results on singular foliations that we require. Section 3 is devoted to the strategy used in the algorithm that checks whether a given polynomial determines a minimal involutive homogeneous hypersurface of $\mathbb{C}^4$. The algorithm itself is described in Section 4, while details of its implementation and application can be found in Section 5. The proof of Theorem 1.2 is the subject of Section 6.

2. *Holomorphic foliations*

Let X be a smooth complex algebraic variety of dimension n. A *one-dimensional foliation* over X is a map $\theta : \Omega^1_X \longrightarrow \mathcal{L}$ from the sheaf of Kähler differentials to some line bundle $\mathcal{L}$ over X. From now on, we will refer to such a map simply as a *foliation of X*.

A *singularity* of θ is a point $p \in X$ at which θ is not surjective. The set of all singular points of θ is an algebraic subvariety of X denoted by $\mathrm{Sing}(\theta)$. From now on, we assume that all the foliations that we consider in this paper have a finite set of singular points. We say that an algebraic subvariety Y of X is *invariant* under θ if there exists a map $\Omega^1_Y \longrightarrow \mathcal{L}|_Y$ such that the following diagram is commutative.

$$\begin{tikzcd}
\Omega^1_X|_Y \arrow[r, "\theta|_Y"] \arrow[d] & \mathcal{L}|_Y \\
\Omega^1_Y \arrow[ur, dotted] &
\end{tikzcd}$$

Given $p \in X$, there exists a neighbourhood U of p with coordinates $x_1, \ldots, x_n$ such that θ is represented on U by a vector field

$$\theta_U = \sum_{i=1}^{n} g_i \, \partial/\partial x_i,$$

where the g_i are regular functions on U. Note that p is a singular point of θ if and only if $\theta_U(p) = 0$. If $p \in \mathrm{Sing}(\theta) \cap U$, we write $J_p(\theta)$ for the 1-jet of θ_U at p. The 1-jet is independent of the choice of local coordinates, and it is equal to the jacobian matrix of the map $U \longrightarrow \mathbb{C}^n$ which sends $q \in U$ to the vector $(g_1(q), \ldots, g_n(q))$. The singularity p is said to be *nondegenerate* if $\det(J_p(\theta)) \neq 0$. The foliation θ is nondegenerate if all its singular points are nondenegerate. The *characteristic exponents* of θ at a nondegenerate singularity p are the ratios λ/λ', where λ and λ' are eigenvalues of $J_p(\theta)$. We say that θ is *of Poincaré type* if all its singularities are nondegenerate and none of its characteristic exponents is a real number. We require this hypothesis in order to use the following consequence of [**14**, Proposition 2.5, p. 656].

PROPOSITION 2.1. *Let θ be a nondegenerate foliation of Poincaré type, and assume that C is an algebraic curve invariant under θ. If C is singular at some $p \in \mathrm{Sing}(\theta)$, then it has at most n smooth analytic branches through p.*

From now on we assume that X has dimension 2. The key result that we use in this paper is a theorem of Suwa's [**19**, Theorem 2.1]. In order to state it, we must define the index $\mathrm{ind}_p(\theta, C)$ of [**19**, p. 2991], where $C \subset X$ is an algebraic curve invariant under θ and $p \in \mathrm{Sing}(\theta) \cap C$. However, instead of giving the definition in full generality, we will do it only for nondegenerate foliations of Poincaré type. If θ is such a foliation, then it follows from Poincaré's theorem [**1**, Chapter 5, §24, p. 187] that the germ of vector field θ_p is biholomorphically equivalent to $\lambda x \, \partial/\partial x + \lambda' y \, \partial/\partial y$, where λ and λ' are the (nonzero) eigenvalues of $J_p(\theta)$ at p. Moreover, the same hypotheses, together with Proposition 2.1, imply that the holomorphic germ of C at p is given in the local coordinate system at p by one of the following three equations: $x = 0$, $y = 0$, or $xy = 0$. Thus, by [**19**, Example 1.6, p. 2992], we find that

$$\mathrm{ind}_p(\theta, C) = \begin{cases} \lambda'/\lambda & \text{if the germ is given by } x = 0; \\ \lambda/\lambda' & \text{if the germ is given by } y = 0; \\ (\lambda' + \lambda)^2/\lambda\lambda' & \text{if the germ is given by } xy = 0. \end{cases}$$

Define

$$S(C, \theta) = \sum_{p \in \mathrm{Sing}(\theta) \cap C} \mathrm{ind}_p(\theta, C).$$

Note that if C is singular at a nondegenerate singularity p of θ, then the formula above gives

$$\operatorname{ind}_p(\theta, C) = \frac{\lambda'}{\lambda} + \frac{\lambda}{\lambda'} + 2.$$

Therefore, if θ is a nondegenerate foliation of Poincaré type and C is an invariant curve with s singular points, then $S(C, \theta) - 2s$ is a sum of characteristic exponents. Applying [19, Theorem 2.1] in this situation, we obtain the following result.

THEOREM 2.2. *Let S be a smooth complex algebraic surface, and let θ be a nondegenerate foliation of Poincaré type on S. If C is a reduced and irreducible algebraic curve of S invariant under θ, and $C \cap \operatorname{Sing}(\theta) \neq \emptyset$, then $C^2 - 2s$ is a sum of characteristic exponents of θ, where s is the number of singularities of C.*

Let $m \geqslant 1$ be an integer, and let $\mathbb{P}^m$ be the complex projective space of dimension m, with homogeneous coordinates $x_0, \ldots, x_m$. We denote by U_j the open set of $\mathbb{P}^m$ defined by $x_j \neq 0$. Given a homogeneous affine variety Y of $\mathbb{C}^{m+1}$, we write $\overline{Y}$ for the projectivization of Y in $\mathbb{P}^m$. In other words, Y is the cone over $\overline{Y}$.

It follows from the Euler exact sequence that a map $\theta : \Omega^1_{\mathbb{P}^m} \longrightarrow \mathcal{O}(k-2)$ is induced by the homogeneous vector field of $\mathbb{C}^{m+1}$ given by $G_0\,\partial/\partial x_0 + \cdots + G_m\,\partial/\partial x_m$, where $G_0, \ldots, G_m$ are homogeneous polynomials of degree $k-1$ in the variables $x_0, \ldots, x_m$. It is easy to see that $\operatorname{Sing}(\theta)$ is the projective variety cut out by the minors of the matrix

$$\begin{bmatrix} x_0 & \cdots & x_m \\ G_0 & \cdots & G_m \end{bmatrix}. \tag{2.1}$$

On the other hand, if Y is the projective subvariety of $\mathbb{P}^m$ determined by the homogeneous radical ideal I, then Y is invariant under θ if and only if

$$G_0\,\partial H/\partial x_0 + \ldots + G_m\,\partial H/\partial x_m$$

belongs to I for every $H \in I$.

A foliation of $\mathbb{P}^m$ determines (and is determined by) a vector field θ_j of U_j. This vector field is obtained by dehomogenizing θ with respect to x_j. It corresponds to the projection of $(\theta - G_j E)|_{x_j=1}$ onto $x_j = 0$, where E is the Euler vector field. Identifying U_j with $\mathbb{C}^m$ in the usual way, we find that

$$\theta_j = \sum_{i \neq j} (G_i - G_j x_i)_{|x_j=1}\,\partial/\partial x_i.$$

It is easy to see that $p \in U_j$ is a singular point of θ if and only if $\theta_j(p) = 0$.

Suppose now that $m = 2n - 1$. We say that the foliation θ is *hamiltonian* if there exists a homogeneous polynomial F of degree k in $x_1, \ldots, x_{2n}$ such that

$$G_i = \begin{cases} \partial F/\partial x_{i+n} & \text{if } 0 \leqslant i \leqslant n; \\ -\partial F/\partial x_{i-n} & \text{if } n+1 \leqslant i \leqslant 2n. \end{cases}$$

We write h_F for this homogeneous vector field.

This foliation is closely related to the symplectic geometry of $\mathbb{C}^{2n}$, as explained in the introduction. Recall that an algebraic variety Y of $\mathbb{C}^{2n}$ is *involutive* if its ideal $\mathrm{I}(Y)$ is closed with respect to the Poisson bracket. In other words, $\{F, G\} \in \mathrm{I}(Y)$, for all $F, G \in \mathrm{I}(X)$. In particular, if Y is contained in the homogeneous hypersurface $\mathcal{Z}(F)$, then

$$h_F(G) = \{F, G\} \in \mathrm{I}(Y) \qquad \text{for all } G \in \mathrm{I}(Y).$$

Therefore, $\overline{Y}$ is a subvariety of $\mathbb{P}^{2n-1}$ invariant under the foliation of $\mathbb{P}^{2n-1}$ induced by the homogeneous vector field h_F. The following elementary property of the involutive varieties is used in Section 3.

PROPOSITION 2.3. *If Y is an involutive variety of $\mathbb{C}^{2n}$, then* $\dim Y \geqslant n$.

3. *The strategy*

In this section we discuss the strategy to be used in proving that a given homogeneous hypersurface of $\mathbb{C}^4$ is a minimal homogeneous involutive variety. We enumerate, along the way, the various hypotheses that are required for the strategy to work. We assume that once a hypothesis has been stated, it will be in force from that point onwards. Of course, the resulting algorithm will have to check each one of these hypotheses before we can be confident that it works correctly.

Let F be a homogeneous irreducible polynomial of degree $k \geqslant 3$ in the variables $x_1, \dots, x_4$, and let $X = \mathcal{Z}(F)$ in $\mathbb{C}^4$. Suppose that Y is an involutive homogeneous subvariety of $\mathbb{C}^4$ contained in X. Let $S = \overline{X}$ and $C = \overline{Y}$. It follows from Proposition 2.3 that $\dim C = 1$. From now on, the following hypothesis will be in force.

HYPOTHESIS 3.1. S is a smooth surface.

Thus C is a curve on S invariant under the foliation θ_F induced by h_F over S. The following lemma will be used often, without further comment.

LEMMA 3.2. *Let F be a homogeneous polynomial of degree $k \geqslant 2$. Assume that $S = \mathcal{Z}(F) \subset \mathbb{P}^3$ is a smooth complex algebraic surface. If $C \subset S$ is a reduced and irreducible algebraic curve invariant under θ_F, then $C \cap \mathrm{Sing}(\theta_F) \neq \emptyset$.*

Proof. Denoting by h_F both the hamiltonian vector field of $\mathbb{C}^4$ and the foliation that it induces on $\mathbb{P}^3$, we see that $\theta_F = (h_F)_{|S}$. In particular, since C is invariant under θ_F, it is also invariant under h_F. But h_F is a foliation of $\mathbb{P}^3$, so it cannot have compact leaves, by [**11**, Proposition 4.2, p. 130]. Therefore, $C \cap \mathrm{Sing}(h_F) \neq \emptyset$. However, by [**3**, Lemma 2, p. 228], we know that $\mathrm{Sing}(h_F) \subset S$. Hence, $C \cap \mathrm{Sing}(\theta_F) \neq \emptyset$, which proves the lemma. □

It follows from Lemma 3.2 that we can apply Theorem 2.2 to θ_F and C. Hence, there is a sum of characteristic exponents of θ_F that is an integer. Therefore, if we could show that there are no integral sums of characteristic exponents of θ_F, then we would conclude that X does not contain any involutive subvarieties. However, the sum of *all* the characteristic exponents of θ_F is always integral. This follows from a famous theorem of Baum and Bott [**2**, Theorem 1, p. 280]. Let λ_p and λ'_p be the eigenvalues of $J_p(\theta_F)$ at a singularity $p \in \mathrm{Sing}(\theta_F)$. Denote by $S(\theta_F)$ the sum of $(\lambda_p + \lambda'_p)^2/\lambda_p\lambda'_p$, for all $p \in \mathrm{Sing}(\theta_F)$. In order to apply the Baum–Bott theorem more easily, we make our second hypothesis.

HYPOTHESIS 3.3. The foliation θ_F induced by h_F on S must be nondegenerate and of Poincaré type.

THEOREM 3.4. *Let F be a homogeneous polynomial of degree $k \geqslant 2$. Assume that $S = \mathcal{Z}(F) \subset \mathbb{P}^3$ is a smooth complex algebraic surface, and that θ_F is a nondegenerate foliation of Poincaré type of S. Then $S(\theta_F) = 4k$.*

Proof. It follows from [2, Theorem 1, p. 280] that

$$S(\theta_F) = \int_S c_1(\Theta_S/\mathcal{O}(2-k))^2.$$

But

$$c_1(\Theta_S) = -c_1\Big(\bigwedge^2 \Theta_S\Big) = -c_1(\mathcal{O}(k-4)).$$

Therefore,

$$c_1(\Theta_S/\mathcal{O}(2-k)) = c_1(\Theta_S) - c_1(\mathcal{O}(2-k)) = 2h,$$

where h is the hyperplane section of S. Since $h^2 = k$, it follows that

$$\int_S c_1(\Theta_S \otimes \mathcal{O}(k-2))^2 = 4k,$$

as required. □

Our next result is also a consequence of the Baum–Bott theorem.

PROPOSITION 3.5. *If θ_F is nondegenerate, then it has*

$$m(k) = (k-1)^3 + (k-1)^2 + (k-1) + 1$$

singular points (*counted with multiplicity*).

Proof. The vector field h_F induces a foliation of degree $k-1$ over $\mathbb{P}^3$. By [14, Remark 4.1, p. 667], this foliation has $m(k)$ singular points. However, by [3, Lemma 2] each one of these singular points belongs to S. Therefore, θ_F has $m(k)$ singular points. □

Combining the last two results, we have the following corollary.

COROLLARY 3.6. *The sum of all the characteristic exponents of θ_F over all its singular points is equal to $-2k^2(k-2)$.*

Proof. Since

$$\frac{(\lambda_p + \lambda'_p)^2}{\lambda_p \lambda'_p} = \frac{\lambda_p}{\lambda'_p} + \frac{\lambda'_p}{\lambda_p} + 2,$$

it follows that the sum of all the characteristic exponents over all the singular points of θ_F is equal to

$$S(\theta_F) - 2m(k) = -2k^2(k-2). \qquad \square$$

This is enough to show that if a curve is invariant under θ_F, then it cannot be singular at all the singularities of θ_F.

PROPOSITION 3.7. *Let $k = 3$ or $k = 4$, and let C be a curve of $S \subset \mathbb{P}^3$ that is invariant under θ_F. Then* $\mathrm{Sing}(C) \subsetneq \mathrm{Sing}(\theta_F)$.

Proof. If $\mathrm{Sing}(C) = \mathrm{Sing}(\theta_F)$, then $S(C, \theta_F) = S(\theta_F)$, and we show that for $k = 3$ and $k = 4$, this leads to a contradiction.

It follows from the genus formula [4, I.15, p. 8] that $C^2 = 2p_a + (4-k)d - 2$, where p_a is the arithmetic genus and d is the degree of C. However, by Proposition 3.5, θ_F has $m(k) = (k-1)^3 + (k-1)^2 + (k-1) + 1$ singularities as a foliation of S. Moreover, by Proposition 2.1, C must have a node at every one of these singularities. Therefore, by [10, Exercise 1.8, p. 298],

$$p_a = g + 2m(k),$$

where g is the genus of the normalization of C. Hence

$$C^2 = 2g + 4m(k) + (4-k)d - 2.$$

But we are assuming that $S(C,\theta) = S(\theta)$. Thus, by Theorems 2.2 and 3.4,

$$C^2 = S(C,\theta) = S(\theta) = 4k.$$

It then follows that

$$2g + (4-k)d = 4k - 4m(k) + 2 = -4k^3 + 8k^2 - 4k + 2.$$

The right-hand side of this equation is negative for all $k \geqslant 3$. Since the left-hand side is positive for $k = 3$ and $k = 4$, we obtain a contradiction in these two cases, and the proposition is proved. □

4. *The algorithm*

In this section we give a step-by-step description of the algorithm whose strategy was discussed in Section 3. We explain what each step does, and what kind of computation has to be performed in order to achieve it. The significance of Step 4 is discussed at the end of this section.

Much of the work done by the algorithm is aimed at checking Hypotheses 3.1 (Step 1) and 3.3 (Steps 5 and 7). Let F be a homogeneous polynomial on x_1, x_2, x_3 and x_4 with rational coefficients. Throughout this section we denote by h_F both the hamiltonian vector field defined by F, and the foliation induced by h_F on $\mathbb{P}^3$, while θ_F is the foliation induced by the vector field h_F on the surface $F = 0$. We also write $m(k) = (k-1)^3 + (k-1)^2 + (k-1) + 1$.

Input: a homogeneous polynomial $F \in \mathbb{Q}[x_1, x_2, x_3, x_4]$, of degree $k \geqslant 3$.

Output: an error message, or

`'The hypersurface defined by` F
`is minimal involutive homogeneous.'`

Step 1 checks that $\overline{\mathcal{Z}(F)}$ is smooth.

Compute the radical of the ideal generated by F and its partial derivatives. If it is not equal to (x_1, x_2, x_3, x_4), print

`'The projective surface is not smooth.'`

and stop.

Step 2 checks that all singularities of h_F belong to U_4.

Compute the radical of the ideal generated by x_4 and the minors of the matrix

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ \partial F/\partial x_3 & \partial F/\partial x_4 & -\partial F/\partial x_1 & -\partial F/\partial x_2 \end{bmatrix}.$$

If it is not equal to (x_1, x_2, x_3, x_4), print

`'There are singularities at infinity.'`

and stop.

Step 3 finds a vector field that determines the foliation h_F in $U_4 \cong \mathbb{C}^3$.
We use x_1, x_2, x_3 to denote the coordinates at U_4. Compute

$$H = (h_F + \partial F/\partial x_2 E)|_{x_4=1},$$

where $E = \sum_{i=1}^4 x_i \, \partial/\partial x_i$ is the Euler vector field. Let h_1, h_2 and h_3 be the coordinate entries of H.

Step 4 checks that the foliation has $m(k)$ distinct singularities.
Compute a Gröbner basis $\{q_i\}$ for the ideal $(h_1, h_2, h_3) \cap \mathbb{Q}[x_i]$, using an elimination order. If one of the q_i is reducible, print

`'The singularities may not be distinct.'`

and stop.

Step 5 checks if there is a degenerate singularity.
Compute a reduced Gröbner basis for the ideal generated by h_1, h_2, h_3 and the determinant of the jacobian matrix $J(H)$ of $H = (h_1, h_2, h_3)$. If it is not $\{1\}$, print

`'There are degenerate singularities.'`

and stop.

Step 6 computes the polynomial of characteristic exponents of the foliation defined by θ_F.
Denote by $M(\lambda)$ the ideal generated by the 3×3 minors of the 4×3 matrix

$$\begin{bmatrix} J(H) - \lambda I \\ \nabla F \end{bmatrix}.$$

Let I be the ideal defined by $h_1, h_2, h_3, \lambda t - \lambda', M(\lambda)$ and $M(\lambda')$. Compute a Gröbner basis G of I with respect to the lexicographic order with

$$\lambda' > \lambda > x_1 > x_2 > x_3 > t,$$

using the FGLM algorithm [7] . Since I is zero-dimensional, it must contain a polynomial in the variable t. Moreover, this polynomial can be written in the form $(t-1)p(t)$ because the zero set of I will always admit solutions with $\lambda = \lambda'$. If $p(t)$ is reducible, print

`'The polynomial of characteristic exponents is reducible.'`

and stop.

Step 7 checks whether the foliation θ_F is of Poincaré type.
Apply Sturm's theorem [6, p. 108] to $p(t)$. If $p(t)$ has real roots, print

`'The foliation is not of Poincaré type.'`

and stop.

Step 8 checks that the polynomial of characteristic exponents has maximum degree.
If $\deg(p(t)) < 2m(k)$, print

`'There are repeated characteristic exponents.'`

and stop.

Step 9 computes the characteristic exponents.
Compute approximations of the roots of $p(t)$ with a sufficiently small error.

Step 10 checks if there is a sum of characteristic exponents that is an integer.
Check if any sum of $2m(k)-1$, or fewer, of these roots is an integer. If there is a sum which is an integer, print

`'There are integral sums of characteristic exponents.'`

and stop; otherwise, print

```
'The hypersurface defined by F
 is minimal involutive homogeneous.'
```

and stop.

We must still discuss the significance of Step 4, especially in its relation to Step 10. The purpose of Step 10 is to show that no sum of indices $\mathrm{ind}_p(\theta_F)$, with p ranging in some proper subset of $\mathrm{Sing}(\theta_F)$, is integral. However, what has actually been implemented is a test to determine if there exists a sum of r *distinct* characteristic exponents that is integral, for some $r < 2m(k)$. Of course, for this strategy to work, the $m(k)$ singularities of θ_F must be distinct, and different singularities must have different characteristic exponents. This is where Step 4 comes to our aid.

First of all, in this step we are checking that a certain ideal is radical. This is done with the help of the following result of Seidenberg [**12**, Proposition 3.7.15, p. 250].

THEOREM 4.1. *Let J be a zero-dimensional radical ideal of $K[x_1, \ldots, x_n]$, where $K \subset \mathbb{C}$ is a field. If, for every $1 \leqslant i \leqslant n$, there exists a nonzero polynomial $g_i \in J \cap K[x_i]$ such that* $\gcd(g_i, dg_i/dx_i) = 1$*, then J is a radical ideal.*

Now it follows from Step 2 that the number of singularities of θ_F (counted with multiplicity) is equal to the dimension of

$$\frac{\mathbb{Q}[x_1, x_2, x_3]}{(h_1, h_2, h_3)}$$

as a vector space over $\mathbb{Q}$. Thus, if (h_1, h_2, h_3) is radical, then all the singularities must be distinct. Therefore, if θ_F passes the test of Step 4, then we can be certain that it has $m(k)$ distinct singularities.

We must now show that the last coordinate of any two distinct points of the set

$$W = \left\{(p, c) \in \mathbb{C}^{n+1} : c \text{ is a characteristic exponent of } \theta_F \text{ at } p \in \mathrm{Sing}(\theta_F)\right\}$$

are distinct. Let I be the ideal defined at Step 6, and denote by $\widetilde{I}$ the ideal generated by I and the polynomial $s(t-1)-1$ in $\mathbb{Q}[x_1, x_2, x_3, x_4, \lambda, \lambda', t, s]$. Let

$$J = \widetilde{I} \cap \mathbb{Q}[x_1, x_2, x_3, x_4, t].$$

Then $W = \mathcal{Z}(J)$. We show the required result using the following lemma. For a proof, see [**12**, Theorem 3.7.25, p. 257].

LEMMA 4.2 (SHAPE LEMMA). *Let I be a zero-dimensional radical ideal of the polynomial ring $K[x_1, \ldots, x_n]$, where $K \subset \mathbb{C}$ is a field. Suppose that the dimension of $K[x_1, \ldots, x_n]/I$ as a vector space over K is equal to the degree of the monic generator g_n of $K[x_n] \cap I$. Then the following statements hold.*

1. *The reduced Gröbner basis of I with respect to the lexicographical order is of the form*

$$\{x_1 - g_1, \ldots, x_{n-1} - g_{n-1}, g_n\}, \qquad g_1, \ldots, g_{n-1} \in K[x_n].$$

2. *The polynomial g_n has $d = \deg g_n$ distinct roots $\alpha_1, \ldots, \alpha_d$ in $\mathbb{C}$, and*

$$\mathcal{Z}(I) = \{(g_1(\alpha_i), \ldots, g_{n-1}(\alpha_i), \alpha_i) : 1 \leqslant i \leqslant d\}.$$

In particular, the last coordinates of any two distinct points of $\mathcal{Z}(I)$ are distinct.

We have already checked (in Steps 4 and 8) that J is a zero-dimensional ideal and that the monic generator of $\mathbb{Q}[t] \cap J$ has the correct dimension. Thus, we need only to show that J is radical, and the required result will hold. However,

$$(g_i) = I \cap \mathbb{Q}[x_i] \subseteq \widetilde{I} \cap \mathbb{Q}[x_i] = J \cap \mathbb{Q}[x_i],$$

for $1 \leqslant i \leqslant 3$. But we have already shown in Step 4 that g_i is irreducible over $\mathbb{Q}$. Therefore, the ideal (g_i) of $\mathbb{Q}[x_i]$ is maximal. Hence, $J \cap \mathbb{Q}[x_i] = (g_i)$ is a prime ideal. Moreover, since

$$(p(t)) \subseteq I \cap \mathbb{Q}[t] \subseteq \widetilde{I} \cap \mathbb{Q}[t] = J \cap \mathbb{Q}[t]$$

and $p(t)$ is irreducible by Step 6, it follows that $J \cap \mathbb{Q}[t]$ is a prime ideal of $\mathbb{Q}[t]$. Therefore, by Seidenberg's lemma, J is a radical ideal of the polynomial ring $\mathbb{Q}[x_1, x_2, x_3, t]$.

5. *Implementation and results*

The algorithm described in the previous section consists of ten steps. The first eight are done symbolically, while the last two steps perform numerical computations in floating-point arithmetic. We implemented the symbolic steps using the computer algebra system SINGULAR (Version 2-0-3) [8]. We also used the numerical library available in SINGULAR to compute the characteristic exponents. However, the SINGULAR program that we wrote to check whether the various sums of characteristic exponents were integers proved to be too slow. This led us to implement this part of the algorithm directly in C. Thus we have split the algorithm of the previous section into three files (see Appendix A).

- `procedures` is implemented in SINGULAR. It contains the procedures that are required to perform the steps described in Section 4.
- `main` is also implemented in SINGULAR. It performs Steps 1 to 9 and returns a file (`roots.txt`) with the real and imaginary parts of the characteristic exponents of θ_F computed to 15 decimal digits.
- `sums` is implemented in C, and corresponds to Step 10. Its input is the file output of `main`, and its output is a file `out.txt`.

Upon receiving its input, `sums` checks that no sum of r characteristic exponents is integral, for $r \leqslant m(k)$. Note that we need not check sums of more than $m(k)$ exponents because we know from Theorem 3.4 that the sum of all the characteristic exponents is an integer. This quite dramatically reduces the time required for computing these sums.

We implemented an algorithm that computes the sums through a number of nested `for` loops. This program assumes that $k = 3$, and that the characteristic exponents are complex numbers whose real part has modulus at most 2.

The real and imaginary parts of the numbers used in `sums` are represented in the type *double*, which guarantees a precision of 15 significant digits. Since at most two digits are enough to represent the integer part of the mantissa of each one of the sums, at least 13 digits remain available for the decimal part. Thus, the absolute error in the representation of each characteristic exponent cannot exceed 10^{-13}. Moreover, we have to sum at most $m(k)$ of these numbers. Therefore, the absolute error for each of these sums cannot exceed $m(k) \cdot 10^{-13}$. For $k = 3$ this gives an error of at most 10^{-11}, which means that we can definitely trust the first 10 decimal digits of the mantissa. The program `sums` takes a number not to be an integer if any of these 10 digits is nonzero, and if they are not all equal to 9.

We now present the results that we obtained by applying the algorithm to the following polynomial of degree 3:

$$\begin{aligned}\mathcal{F} = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1^2x_4 + 2x_1x_2^2 + 2x_1x_3^2 + 2x_1x_4^2 + 8x_2^3 + x_2^2x_3 \\ + x_2^2x_4 + 2x_2x_3^2 + 2x_2x_4^2 + x_3^3 + x_3^2x_4 + 2x_3x_4^2 + 8x_4^3.\end{aligned}$$

The characteristic exponents of $\theta_{\mathcal{F}}$ are the roots of an irreducible one-variable polynomial of degree 30, namely

$$\begin{aligned}
p(t) = {}&\\
&1\,424\,796\,099\,432\,013\,162\,078\,686\,196\,898\,890\,282\,091\,710\,120\,981\,672\,625\,009(t^{30}+1)\\
&+25\,646\,329\,789\,776\,236\,917\,416\,351\,544\,180\,025\,077\,650\,782\,177\,670\,107\,250\,162(t^{29}+t)\\
&+208\,719\,751\,477\,278\,207\,557\,068\,054\,763\,472\,255\,012\,672\,013\,172\,760\,985\,302\,947(t^{28}+t^2)\\
&+993\,774\,241\,989\,011\,452\,724\,494\,253\,583\,473\,446\,478\,329\,465\,627\,721\,741\,527\,300(t^{27}+t^3)\\
&+2\,934\,151\,927\,851\,069\,813\,467\,768\,496\,725\,417\,790\,099\,211\,662\,395\,946\,239\,595\,997(t^{26}+t^4)\\
&+4\,850\,559\,593\,996\,588\,654\,984\,770\,830\,359\,044\,659\,161\,269\,643\,905\,711\,653\,692\,422(t^{25}+t^5)\\
&+785\,945\,048\,465\,657\,668\,604\,960\,376\,476\,295\,275\,969\,394\,851\,004\,593\,754\,434\,495(t^{24}+t^6)\\
&-17\,477\,009\,553\,027\,230\,339\,989\,135\,355\,513\,126\,370\,533\,241\,703\,776\,928\,850\,722\,712(t^{23}+t^7)\\
&-44\,504\,853\,905\,255\,806\,049\,333\,072\,942\,045\,220\,211\,232\,255\,933\,116\,231\,668\,230\,803(t^{22}+t^8)\\
&-45\,588\,527\,721\,498\,682\,414\,532\,196\,654\,071\,923\,958\,937\,169\,383\,481\,123\,689\,694\,654(t^{21}+t^9)\\
&+19\,340\,938\,917\,972\,092\,540\,387\,438\,832\,586\,947\,514\,732\,886\,107\,772\,044\,177\,203\,015(t^{20}+t^{10})\\
&+127\,905\,471\,034\,714\,231\,098\,198\,821\,850\,282\,677\,184\,610\,726\,609\,434\,588\,916\,322\,620(t^{19}+t^{11})\\
&+170\,932\,571\,090\,857\,366\,892\,573\,428\,984\,877\,226\,995\,591\,602\,951\,724\,167\,485\,830\,009(t^{18}+t^{12})\\
&+56\,489\,617\,051\,266\,221\,694\,646\,175\,317\,820\,063\,864\,861\,238\,308\,999\,527\,879\,984\,070(t^{17}+t^{13})\\
&-149\,550\,394\,434\,265\,981\,523\,928\,562\,114\,623\,689\,697\,009\,464\,850\,343\,740\,260\,328\,669(t^{16}+t^{14})\\
&-254\,102\,055\,569\,328\,976\,647\,969\,237\,196\,536\,931\,854\,556\,092\,369\,333\,450\,743\,854\,416t^{15}.
\end{aligned}$$

Note that the sum of the roots of this polynomial is

$$-\frac{25\,646\,329\,789\,776\,236\,917\,416\,351\,544\,180\,025\,077\,650\,782\,177\,670\,107\,250\,162}{1\,424\,796\,099\,432\,013\,162\,078\,686\,196\,898\,890\,282\,091\,710\,120\,981\,672\,625\,009} = -18,$$

as expected from Corollary 3.6. None of the roots of this polynomial is a real number.

Listing only one root of each pair of complex conjugate roots to 15 decimal digits, we have:

$$
\begin{aligned}
-0.493\,604\,660\,708\,982 + 0.118\,269\,412\,246\,561i;\\
-0.578\,203\,943\,097\,618 + 0.255\,690\,449\,211\,412i;\\
-0.582\,233\,372\,173\,013 + 0.372\,905\,625\,105\,277i;\\
-0.602\,557\,103\,820\,076 + 0.337\,250\,062\,983\,222i;\\
0.721\,791\,882\,015\,353 + 0.025\,999\,147\,890\,059i;\\
-0.786\,553\,781\,560\,882 + 0.060\,252\,640\,255\,812i;\\
-0.934\,831\,035\,045\,717 - 0.209\,948\,982\,884\,531i;\\
0.999\,012\,562\,172\,831 + 0.044\,428\,601\,383\,287i;\\
-1.018\,348\,172\,483\,035 + 0.228\,705\,675\,164\,794i;\\
-1.217\,923\,081\,533\,882 + 0.780\,048\,670\,783\,815i;\\
-1.263\,718\,738\,660\,996 + 0.707\,300\,970\,322\,240i;\\
-1.263\,951\,920\,695\,255 + 0.096\,822\,928\,277\,282i;\\
1.383\,645\,681\,417\,165 + 0.049\,839\,309\,079\,181i;\\
-1.446\,604\,193\,449\,767 + 0.639\,710\,054\,678\,468i;\\
-1.915\,920\,122\,376\,120 + 0.459\,061\,197\,800\,113i.
\end{aligned}
$$

No sum of 15, or fewer, numbers chosen from among these roots and their complex conjugates is an integer. Thus the involutive homogeneous hypersurface $\mathcal{Z}(\mathcal{F})$ must be minimal.

Running under Windows Me on a PC with a Pentium III processor at 1.0 GHz, the program `main` took 211 seconds to produce the list of roots given above, while `sums` returned its verdict within 696 seconds.

6. *Proof of Theorem* 1.2

In this section we prove Theorem 1.2. Throughout the section we assume that there exists a homogeneous polynomial $\mathcal{F} \in \mathbb{C}[x_1, x_2, x_3, x_4]$, of degree $k \geqslant 3$, such that

- $\overline{\mathcal{Z}(\mathcal{F})}$ is smooth;
- $\theta_{\mathcal{F}}$ is nondegenerate of Poincaré type;
- $\theta_{\mathcal{F}}$ has $m(k) = (k-1)^3 + (k-1)^2 + (k-1) + 1$ distinct singularities;
- $\theta_{\mathcal{F}}$ has $2m(k)$ distinct characteristic exponents;
- x_1^k has nonzero coefficient in $\mathcal{F}$;
- no sum of r characteristic exponents of $\theta_{\mathcal{F}}$ is an integer, for any $r < m(k)$.

For $k = 3$, we can take $\mathcal{F}$ to be the polynomial displayed in Section 5.

As in Section 1, we identify $\mathcal{S}(4, k)$ with $\mathbb{A}^{N(k)}$, where $N(k) = \binom{4+k}{k}$. Let $[F]$ be the class of $F \in \mathbb{A}^{N(k)}$ in the projective space $\mathbb{P}^{N(k)-1}$, and write

$$D_F = h_F + \frac{\partial F}{\partial x_2} E = (A_1, A_2, A_3, A_4),$$

where E is the Euler vector field.

Let

$$G : \mathbb{C}^{N(k)} \times \mathbb{C}^4 \longrightarrow \mathbb{C}^3$$

be the map defined by $G(F, p) = (A_1(p), A_2(p), A_3(p))$, and denote by $J(F)$ the jacobian of G with respect to x_1, x_2 and x_3. Consider the matrices

$$A = \begin{bmatrix} J(F) - \lambda I \\ \nabla F \end{bmatrix}$$

and

$$B = \begin{bmatrix} \partial F/\partial x_3 & \partial F/\partial x_4 & -\partial F/\partial x_1 & -\partial F/\partial x_2 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix}.$$

Writing $M(C)$ for the ideal of maximal minors of a matrix C, let

$$J = (F) + M(A) + M(B) \qquad \text{and} \qquad \mathcal{X} = \mathcal{Z}(J) \subseteq \mathbb{P}^{N(k)} \times \mathbb{P}^3.$$

If $p \in U_4$, then $([F : \lambda], p) \in \mathcal{X}$ if and only if

- p is a singularity of θ_F, and
- λ is an eigenvalue of θ_F at p.

Moreover, if $([F : \lambda], p) \in \mathcal{X}$ and $F = 0$, then $\lambda = 0$. Thus, there exists a well-defined map

$$\pi : \mathbb{P}^{N(k)} \times \mathbb{P}^3 \longrightarrow \mathbb{P}^{N(k)-1}$$

given by $\pi([F : \lambda], p) = [F]$. By [17, Corollary, p. 116], the set

$$Y_1 = \left\{[F] \in \mathbb{P}^{N(k)-1} : \dim \pi^{-1}([F]) \geqslant 1\right\}$$

is closed in $\mathbb{P}^{N(k)-1}$. Since $\mathcal{F} \notin Y_1$, it follows that Y_1 is a proper subset of $\mathbb{P}^{N(k)-1}$. Therefore, $\dim Y_1 < N(k) - 1$.

The set

$$Y_2 = \pi\left(\{([F : \lambda], p) : \lambda \cdot x_4 \cdot \partial F/\partial x_j = 0 \text{ for all } 1 \leqslant j \leqslant 4\}\right)$$

is also a proper closed subset of $\mathbb{P}^{N(k)-1}$ because $\mathcal{F} \notin Y_2$. Now take $U = \mathbb{P}^{N(k)-1} \backslash (Y_1 \cup Y_2)$. Since $\pi^{-1}[F]$ is finite for every $[F] \in U$, it follows from [9, Lemma 14.8, p. 178] that the map

$$\pi_{|U} : \pi^{-1}(U) \longrightarrow U,$$

obtained by restricting π to $\pi^{-1}(U)$, is finite.

Moreover, $\sharp\pi^{-1}([F]) \leqslant 2m(k)$ for every $[F] \in U$. Since $\sharp\pi^{-1}([\mathcal{F}]) = 2m(k)$, then by [17, Theorem 7, p. 116] the set

$$V = \{[F] \in U : \sharp\pi^{-1}([F]) = 2m(k)\} \neq \emptyset$$

is open in U. Therefore, if $[F] \in V$, then $([F : \lambda], p) \in \mathcal{X}$ satisfies the following conditions:

1. $F = 0$ is a smooth surface of $\mathbb{P}^3$;
2. θ_F is nondegenerate at every one of its singularities;
3. all singularities of θ_F are distinct; and
4. the foliation θ_F has two distinct eigenvalues at each one of its singularities.

Furthermore, $\mathcal{F} \in V$.

Let V_0 be the open subset of V of those polynomials for which the coefficient of x_1^k is nonzero. Since $V_0 \neq \emptyset$, it is dense in $\mathbb{P}^{N(k)-1}$. We may identify V_0 with an open subset of $\mathbb{A}^{N(k)-1}$. Moreover, since V_0 is an open nonempty set in the Zariski topology, it is also a dense open set in the analytic topology. Thus $G_{|V_0}$ gives rise to a function $G_0 : V_0 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^3$.

Choose a polynomial $F_0 \in V_0$ such that $\mathrm{Sing}(h_{F_0}) = \{p_1, \ldots, p_{m(k)}\}$. Since h_{F_0} is nondegenerate at $p_j \in \mathrm{Sing}(h_{F_0})$, it follows that the jacobian of G_0 with respect to the coordinates of $\mathbb{C}^3$ is nonzero. Thus, by the implicit function theorem, there exist open analytic neighbourhoods $\mathcal{U}(F_0)$ of F_0 and W_i of p_i, and functions $\psi_i : \mathcal{U}(F_0) \longrightarrow W_i$, such that

$$G_0(F, \psi_i(F)) = 0 \qquad \text{for all } F \in \mathcal{U}(F_0).$$

Let $F \in \mathcal{U}(F_0)$ and $1 \leqslant i \leqslant m(k)$. Denote by $J_i(F)$ the 1-jet of θ_F at $\psi_i(F)$. If q and $1/q$ are the characteristic exponents of θ_F at $\psi_i(F)$, then

$$(\det J_i(F))q^2 + (2 \det J_i(F) - (\mathrm{tr}\, J_i(F)^2)q + \det J_i(F) = 0. \tag{6.1}$$

Since the eigenvalues of θ_F at $\psi_i(F)$ are distinct, it follows that the discriminant of this equation is nonzero. Hence, by shrinking $\mathcal{U}(F_0)$ if necessary, we can construct a holomorphic function $\rho_i : \mathcal{U}(F_0) \longrightarrow \mathbb{C}$ such that $\rho_i(F)$ satisfies (6.1), for every $F \in \mathcal{U}(F_0)$.

We will define Ω locally at F_0 as follows. Given a subset S of $\mathcal{S}_k = \{1, \ldots, m(k)\}$ and $F \in \mathcal{U}(F_0)$, set

$$\rho_S(F) = \sum_{i \in S} \rho_i(F).$$

Since $\mathbb{Z}$ is a closed subset of $\mathbb{C}$ in the analytic topology, it follows that $\rho_S^{-1}(\mathbb{Z})$ is a closed subset of $\mathcal{U}(F_0)$ in the topology induced from the analytic topology of $\mathbb{A}^{N(k)-1}$. Now

$$\Omega \cap \mathcal{U}(F_0) = \mathcal{U}(F_0) \setminus \bigcup_{S \subsetneq \mathcal{S}_k} \rho_S^{-1}(\mathbb{Z}).$$

Note that if $\Omega \cap \mathcal{U}(F_0)$ is nonempty, then it must be dense in $\mathcal{U}(F_0)$. Moreover, $\mathcal{F} \in \Omega \cap \mathcal{U}(\mathcal{F}) \neq \emptyset$. Since V_0 is connected, it follows that Ω must be be a dense nonempty set of V_0, and so also of $\mathbb{P}^{N(k)-1}$.

Acknowledgements. We thank I. Vainsencher, Mauro Rincón and the referee for their many helpful suggestions.

Appendix A. *Program files*

This appendix contains the files `procedures`, `main` and `sums`, and can be found at

`http://www.lms.ac.uk/jcm/8/lms2003-033/appendix-a`.

See the `README.txt` file included there for an explanation of how to use the programs.

The files are also available for downloading from

`http://www.dcc.ufrj.br/~collier/folia.html`.

References

1. V. I. Arnold, *Geometrical methods in the theory of ordinary differential equations* (Springer, 1983). 303
2. P. Baum and R. Bott, 'Singularities of holomorphic foliations', *J. Differential Geom.* 7 (1972) 279–342. 305, 306
3. I. N. Bernstein and V. Lunts 'On non-holonomic irreducible D-modules', *Invent. Math.* 94 (1988) 223–243. 301, 302, 305, 306

4. A. Beauville, *Complex algebraic surfaces*, 2nd edn (Cambridge University Press, 1996). 306

5. N. Chriss and V. Ginzburg, *Representation theory and complex geometry* (Birhäuser, Boston, 1997). 301

6. J. H. Davenport, Y. Siret and E. Tournier, *Computer algebra* (Academic Press, London, 1988). 308

7. J. C. Faugère, P. Gianni, D. Lazard and T. Mora, 'Efficient computation of zero-dimensional Gröbner bases by change of ordering', *J. Symbolic Comput.* 16 (1993) 329–344. 308

8. G.-M. Greuel, G. Pfister, and H. Schönemann, 'Singular 2.0. A computer algebra system for polynomial computations', Centre for Computer Algebra, University of Kaiserslautern (2001), `http://www.singular.uni-kl.de`. 310

9. J. D. Harris, *Algebraic geometry: a first course* (Springer, 1993). 313

10. R. Hartshorne, *Algebraic geometry*, Grad. Texts in Math. 52 (Springer, 1977). 307

11. J. P. Jouanolou, *Equations de Pfaff algébriques*, Lect. Notes in Math. 708 (Springer, 1979). 302, 305

12. M. Kreuzer and L. Robbiano, *Computational commutative algebra 1* (Springer, Berlin/Heidelberg, 2000). 309

13. A. Lins Neto, *Algebraic solutions of polynomial differential equations and foliations in dimension two*, Lect. Notes in Math. 1345 (1988) 192–232. 302

14. A. Lins Neto and M. G. Soares, 'Algebraic solutions of one dimensional foliations', *J. Differential Geom.* 43 (1996) 652–673. 302, 303, 306

15. V. Lunts, 'Algebraic varieties preserved by generic flows', *Duke Math. J.* 58 (1989) 531–554. 302

16. T. McCune, 'A generic cubic surface contains no involutive curve', *J. Algebra* 235 (2001) 36–44. 302

17. I. Shafarevich, *Basic algebraic geometry* (Springer, Heidelberg, 1977). 313

18. M. G. Soares, 'On algebraic sets invariant by one-dimensional foliations on $\mathbf{CP}(3)$', *Ann. Inst. Fourier* 43 (1993) 143–162. 302

19. T. Suwa, 'Indices of holomorphic vector fields relative to invariant curves on surfaces', *Proc. Amer. Math. Soc.* 123 (1995) 2989–2997. 302, 303, 304

L. C. O. Almeida `lcoalmeida@yahoo.com`
S. C. Coutinho `collier@impa.br`
`http://www.dcc.ufrj.br/~collier`

Departamento de Ciência da Computação
Instituto de Matemática
Universidade Federal do Rio de Janeiro
P.O. Box 68530
21945-970 Rio de Janeiro, RJ
Brazil