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d-Simple rings and simple $\mathscr{D}$-modules

By S. C. COUTINHO
Instituto de Matemática, Universidade Federal do Rio de Janeiro,
P.O. Box 68530, 21945-970 *Rio de Janeiro, Brazil.*
e-mail: collier@impa.br

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Abstract

We use certain derivations of a polynomial ring, that do not leave any proper non-zero ideal invariant, to construct simple non-holonomic modules over the nth Weyl algebra. This approach extends to the rings of differential operators of other smooth affine varieties, like smooth quadric surfaces.

1. *Introduction*

It was believed for some time that all simple (irreducible) modules over a Weyl algebra A_n were holonomic; that is, modules of Gelfand–Kirillov dimension n. In 1985, Stafford in [**Sta**] constructed examples of non-holonomic simple A_n-modules. His approach was to write down an element $d \in A_n$, for $n \geqslant 2$, and show that A_n/A_nd is simple by a direct calculation.

A second approach was introduced by Bernstein and Lunts in 1988, (see [**BeL**]). Their starting point was a generic object: either an operator with generic symbol, if one works under the Bernstein filtration, or a generic derivation, if one works under the order filtration. This gives rise to a method that yields many interesting families of non-holonomic simple A_n-modules. However, Stafford's original example is not a member of any of these families. The aim of this paper is to construct a new family of simple non-holonomic modules which contains this example.

Let d be a derivation of a commutative domain R. An ideal I of R is a d-ideal if $d(I) \subseteq I$. We will say that R is d-simple if it does not contain any proper non-zero d-ideals. Commutative d-simple rings have been used in the construction of examples of simple non-commutative rings; see [**GoW**, proposition 1·14].

It is well known that there exist derivations d with respect to which the ring of polynomials in n variables over a field of characteristic zero is d-simple. In Section 3 we show that for some of these derivations a polynomial f can be found such that $A_n/A_n(d+f)$ is a simple A_n-module. Stafford's example can be put into this form. This approach has the advantage that it can be easily generalised to the ring of differential operators of other smooth affine varieties.

The paper is planned as follows. In Section 2 we review some basic facts about $\mathscr{D}$-modules. In this section we also prove the theorems that will allow us to reduce the construction of simple non-holonomic modules over a ring of differential operators of certain smooth varieties to the same problem over the Weyl algebra. The latter is dealt with in Section 3. These results are applied to smooth quadric surfaces in Section 4.

2. *Basic results*

Let us review some basic facts about rings of differential operators and their modules. Let X be an irreducible, smooth, affine variety of dimension n over a field K of characteristic zero. Let R be a localization of the ring of coordinates $\mathcal{O}(X)$ on a multiplicative system. The ring of differential operators $\mathcal{D}(R)$ is the K-subalgebra of $\mathrm{End}_K R$ generated by R and the derivations in $\mathrm{Der}_K R$. This is a simple noetherian domain see [**McR**, chapter 15]. As usual, we will write $\mathcal{D}(X)$ instead of $\mathcal{D}(\mathcal{O}(X))$.

The ring $\mathcal{D}(R)$ admits a filtration, defined by $\mathcal{C}_0 = R$, $\mathcal{C}_1 = R + \mathrm{Der}_K R$ and $\mathcal{C}_k = \mathcal{C}_1^k$ if $k > 0$. The order of a non-zero operator $D \in \mathcal{D}(R)$ is the smallest k such that $D \in \mathcal{C}_k$. It follows from [**McR**, proposition 15·4·5] that the graded ring associated with this filtration is isomorphic to the symmetric algebra on $\mathrm{Der}_K R$. We will denote this algebra by $\mathrm{S}(R)$; or $\mathrm{S}(X)$ if $R = \mathcal{O}(X)$. Let $\mathrm{S}^k(R)$ be the k-th homogenous component of the symmetric algebra. The symbol map of order k, denoted by σ_k, is defined by the composition

$$\sigma_k\colon \mathcal{C}_k \to \mathcal{C}_k/\mathcal{C}_{k-1} \xrightarrow{\simeq} \mathrm{S}^k(R).$$

If $d \in \mathcal{C}_k \setminus \mathcal{C}_{k-1}$ then its principal symbol is $\sigma(d) = \sigma_k(d)$.

The algebra $\mathrm{S}(R)$ has an additional structure of Lie algebra. Let f_1 and f_2 be homogeneous elements of $\mathrm{S}(R)$ of degrees r_1 and r_2. There exist $a_1, a_2 \in \mathcal{D}(R)$ of orders r_1 and r_2 respectively, such that $\sigma_{r_i}(a_i) = f_i$. The *Poisson bracket* of f_1 and f_2 is defined by

$$\{f_1, f_2\} = \sigma_{r_1+r_2-1}([a_1, a_2])$$

where $[a_1, a_2]$ denotes the commutator in $\mathcal{D}(R)$. This is easily extended, by linearity, to all of $\mathrm{S}(R)$. An ideal of $\mathrm{S}(R)$ is involutive if it is closed under the Poisson bracket.

Given a finitely generated left $\mathcal{D}(R)$-module M, we can endow it with a filtration as follows. Let $u_1, \ldots, u_r$ be generators of M. The filtration is defined by $\Gamma_j = \sum_{i=1}^{r} \mathcal{C}_j u_i$ for $j \geqslant 0$. This is a good filtration in the sense that the associated graded module $\mathrm{gr}^\Gamma M$ is finitely generated over $\mathrm{S}(R)$. The radical of the annihilator of $\mathrm{gr}^\Gamma M$ in $\mathrm{S}(R)$ is independent of the good filtration of M used in calculating it, see [**McR**, proposition 8·6·17]. It is called the characteristic ideal of M and denoted by $\mathrm{I}(M)$. The most important property of $\mathrm{I}(M)$ is its involutivity with respect to the Poisson bracket. This is proved in [**Gab**].

A simple, but useful, example occurs when M is cyclic. In this case $M \cong \mathcal{D}(R)/J$, where J is a left ideal of $\mathcal{D}(R)$ and $\mathrm{I}(M)$ is the radical of the symbol ideal of J, which is $\sigma(J) = \sum_{k \geqslant 0} \sigma_k(J \cap \mathcal{C}_k)$.

For the remainder of the section we shall assume that K is algebraically closed. The dimension of a finitely generated left $\mathcal{D}(R)$-module M is defined as the Krull dimension of $\mathrm{S}(R)/\mathrm{I}(M)$. We will denote the dimension of M by $\mathrm{d}_{\mathcal{D}(R)}(M)$; or simply $\mathrm{d}(M)$. A good filtration of M induces a good filtration in a submodule N of M, see [**Cou**, lemma 7·5·1]. Moreover, the dimension defined above is exact:

$$\mathrm{d}(M) = \max\{\mathrm{d}(N), \mathrm{d}(M/N)\};$$

see [**MeN**, section 1·1]. Note that if $a \in \mathscr{D}(X)$ has order $\geqslant 1$ then it follows immediately from the definition that the module $\mathscr{D}(X)/\mathscr{D}(X)a$ has dimension $2n-1$. If $n \geqslant 2$ then $2n-1 > n$ and this module is not holonomic.

We shall now turn to the problem of constructing simple $\mathscr{D}(X)$-modules. The key ingredient will be the fact that $\mathscr{O}(X)$ is d-simple with respect to a derivation d.

THEOREM 2·1. *Let X be an irreducible, smooth, affine variety over K. Suppose that there exists a derivation d of $\mathscr{O}(X)$ with respect to which this ring is d-simple. Let $\mathscr{S} \subseteq \mathscr{O}(X)\backslash\{0\}$ be a multiplicative set and put $M = \mathscr{D}(X)/\mathscr{D}(X)(d+f)$, where $f \in \mathscr{O}(X)$. Thus:*

(1) *if N is a non-zero submodule of M then $N_{\mathscr{S}}$ is a non-zero submodule of $M_{\mathscr{S}}$;*

(2) *if $M_{\mathscr{S}}$ is a simple $\mathscr{D}(X)_{\mathscr{S}}$-module then M is a simple $\mathscr{D}(X)$-module.*

Proof. Suppose that J is a left ideal of $\mathscr{D}(X)$ which contains $\mathscr{D}(X)(d+f)$ properly. To prove (1) it is enough to show that

$$\mathscr{D}(X)_{\mathscr{S}}(d+f) \subsetneqq J_{\mathscr{S}}.$$

Assume, by contradiction, that these ideals are equal after the localization has been performed. Let a be the non-zero element of smallest possible order in $J\backslash\mathscr{D}(X)(d+f)$. The equality of the localizations implies that there exists $s \in \mathscr{S}$ such that $sa \in \mathscr{D}(X)(d+f)$. Taking symbols, we have that $s\sigma(a)$ belongs to the ideal of $S(X)$ generated by $\sigma(d)$. Since $\mathscr{O}(X)$ is d-simple and d has order 1, it follows that $\sigma(d)$ is irreducible. Therefore either s or $\sigma(a)$ belongs to $S(X)\sigma(d)$. But if $\sigma(a) = \sigma(b)\sigma(d)$ for some $b \in \mathscr{D}(X)$, then $a - b(d+f) \in J$ has smaller order than a. Thus $a-b(d+f) \in \mathscr{D}(X)(d+f)$ and so $a \in \mathscr{D}(X)(d+f)$, a contradiction. Hence $s \in S(X)\sigma(d)$, which is not possible either, because $S(X)$ is graded and s has order 0, whilst $\sigma(d)$ has order 1. This proves (1).

To prove (2) we must show that $\mathscr{D}(X)(d+f)$ is a maximal left ideal. Let J be as above; then $\mathscr{D}(X)/J$ is a homomorphic image of M. Since $M_{\mathscr{S}}$ is simple and localization is an exact functor, it follows from (1) that $J_{\mathscr{S}} = \mathscr{D}(X)_{\mathscr{S}}$. Thus $\mathscr{S} \cap J \neq \emptyset$. In particular, $\mathscr{O}(X) \cap J$ is a non-zero ideal of $\mathscr{O}(X)$. But $d+f \in J$, so if $a \in \mathscr{O}(X) \cap J$, then

$$[d+f, a] = d(a) \in \mathscr{O}(X) \cap J.$$

Thus $\mathscr{O}(X) \cap J$ is a non-zero d-ideal of $\mathscr{O}(X)$. Since $\mathscr{O}(X)$ is d-simple, we conclude that $1 \in J$. Hence $\mathscr{D}(X)(d+f)$ is a maximal left ideal, as we wanted to prove.

We will say that X is a select variety if it is an irreducible affine variety whose module of Kähler differentials $\boldsymbol{\Omega}^1(X)$ is free on $dx_1, \ldots, dx_n$ where $x_1, \ldots, x_n \in \mathscr{O}(X)$. The functions $x_1, \ldots, x_n$ will be called the coordinates of X. It is shown in [**McR**, theorem 15·2·13] that every affine, irreducible, smooth variety admits a finite cover by principal open sets which are select varieties. If X is a select variety then there exist derivations ∂_i such that $\partial_i(x_j) = \delta_{ij}$, for $i = 1, \ldots, n$. The set $\{\partial_1, \ldots, \partial_n\}$ is a basis for the module of derivations of X. Put $K[\boldsymbol{x}] = K[x_1, \ldots, x_n]$ and let $K(\boldsymbol{x})$ be its quotient ring. Since $K[\boldsymbol{x}]$ is a polynomial ring, the ring of differential operators $\mathscr{D}(X)$ contains a copy of the nth Weyl algebra, generated by the xs and ∂s, which will be denoted by A_n. Moreover, the quotient field $K(X)$ of $\mathscr{O}(X)$ is a finite extension of $K(\boldsymbol{x})$. Note also that a derivation d of $K[\boldsymbol{x}]$ extends to a derivation of $\mathscr{O}(X)$, which will also be denoted by d.

Theorem 2·2. *Let X be a select variety and let d be a derivation of $K[\boldsymbol{x}]$. Suppose that $K[\boldsymbol{x}]$ is d-simple, then:*

(1) *$\mathcal{O}(X)$ is d-simple;*

(2) *if for some $f \in K[\boldsymbol{x}]$ the module $A_n/A_n(d+f)$ is simple, then so is $\mathcal{D}(X)/\mathcal{D}(X)(d+f)$.*

Proof. Let $\mathscr{S} = K[\boldsymbol{x}] \setminus \{0\}$. It follows from the comments above that $\mathcal{O}(X)_{\mathscr{S}} = K(X)$. For the purposes of this proof it is convenient to write L for $K(X)$ and L_0 for $K(\boldsymbol{x})$. Note that $\mathcal{D}(X)_{\mathscr{S}} = \mathcal{D}(L)$ and $S(X)_{\mathscr{S}} = S(L)$.

We begin by proving (1). Let J be a non-zero d-ideal of $\mathcal{O}(X)$. Since $J_{\mathscr{S}} = K(X)$, it follows that $J \cap \mathscr{S} \neq \emptyset$. In particular, the d-ideal $J \cap K[\boldsymbol{x}]$ of $K[\boldsymbol{x}]$ is non-zero. But $K[\boldsymbol{x}]$ is d-simple, so $1 \in J \cap K[\boldsymbol{x}]$. Hence $1 \in J$. We have thus proved that $\mathcal{O}(X)$ is d-simple.

Next we turn to (2). Suppose that J is a left ideal of $\mathcal{D}(X)$ such that $(d+f) \notin J$. We claim that it is enough to show that $J_{\mathscr{S}} = \mathcal{D}(L)$. Indeed, that means that there exists a non-zero element $g \in K[\boldsymbol{x}] \cap J$. But $d + f \in J$, so

$$[d+f, g] = d(g) \in J \cap K[\boldsymbol{x}].$$

Since $K[\boldsymbol{x}]$ is d-simple, it follows that $1 \in J \cap K[\boldsymbol{x}] \subseteq J$, as required.

Let us now prove that $J_{\mathscr{S}} = \mathcal{D}(L)$. Localizing $\mathcal{D}(X)(d+f) \subsetneqq J$ at $\mathscr{S}$ and using Theorem 2·1(1), we have that

$$\mathcal{D}(L)(d+f) \subsetneqq J_{\mathscr{S}}. \tag{2·3}$$

To simplify the notation write $J' = J_{\mathscr{S}}$. Taking symbols on (2·3) we get that

$$S(L)\sigma(d) \subsetneqq \sigma(J').$$

But $\sigma(d)$ is irreducible in $S(L)$, thus the Krull dimension of $S(L)/\sigma(J')$ satisfies the inequality

$$\dim\,(S(L)/\sigma(J')) < 2n - 1. \tag{2·4}$$

Since L is a finite extension of L_0 we conclude that $S(L)$ is an integral extension of $S(L_0)$. Thus so is the extension

$$\frac{S(L_0)}{S(L_0) \cap \sigma(J')} \hookrightarrow \frac{S(L)}{\sigma(J')}.$$

From this and (2·4) it follows that

$$\dim\left(\frac{S(L_0)}{S(L_0) \cap \sigma(J')}\right) = \dim\left(\frac{S(L)}{\sigma(J')}\right) < 2n - 1. \tag{2·5}$$

On the other hand, $\mathcal{D}(L)/J'$ is a finitely generated module over $\mathcal{D}(L_0)$, since L is finite over L_0. Moreover the filtration induced by the order of $\mathcal{D}(L)$ is a good filtration of $\mathcal{D}(L)/J'$ as a $\mathcal{D}(L_0)$-module. The corresponding graded module is $S(L)/\sigma(J')$. Thus

$$\mathrm{ann}_{S(L_0)}\left(\frac{S(L)}{\sigma(J')}\right) = \sigma(J') \cap S(L_0).$$

In particular

$$\mathrm{d}_{\mathcal{D}(L_0)}(\mathcal{D}(L)/J') = \dim\,(S(L_0)/\sigma(J') \cap S(L_0)) < 2n - 1,$$

by (2·5). Since the natural map

$$\frac{\mathscr{D}(L_0)}{\mathscr{D}(L_0) \cap J'} \hookrightarrow \frac{\mathscr{D}(L)}{J'}$$

is a homomorphism of finitely generated $\mathscr{D}(L_0)$-modules, we conclude that

$$\mathrm{d}_{\mathscr{D}(L_0)}(\mathscr{D}(L_0)/(\mathscr{D}(L_0) \cap J')) < 2n - 1.$$

Hence $\mathscr{D}(L_0)(d + f) \subsetneqq \mathscr{D}(L_0) \cap J'$. But $A_n/A_n(d + f)$ is a simple A_n-module by hypothesis. Thus its localization at $\mathscr{S}$ is a simple $\mathscr{D}(L_0)$-module; but this module is $\mathscr{D}(L_0)/\mathscr{D}(L_0)(d + f)$. In other words, $\mathscr{D}(L_0)(d + f)$ is a maximal left ideal of $\mathscr{D}(L_0)$. Therefore $1 \in J'$, as we wanted to prove.

We are now ready to outline the strategy we will use to construct non-holonomic simple $\mathscr{D}$-modules. Suppose that X is a smooth, irreducible, affine complex variety. The first step is to construct a derivation d of $\mathscr{O}(X)$ with respect to which this ring is d-simple. This is not always possible. It is easy to see that if such a derivation exists then $\mathbf{\Omega}^1(X)$ has a free direct summand. For example, if X is a surface in $\mathbb{A}^2(K)$, the d-simplicity of $\mathscr{O}(X)$ implies that $\mathbf{\Omega}^1(X)$ is free; see [**Arc**, theorem 2·5·18]. No easy necessary and sufficient condition is known for checking that $\mathscr{O}(X)$ is d-simple for a given X.

Let us assume that a derivation d has been found such that $\mathscr{O}(X)$ is d-simple. The second step consists in finding $f, g \in \mathscr{O}(X)$ such that:

(1) the principal open set U of X defined by $g \neq 0$ is a select variety with coordinates $x_1, \ldots, x_n$;

(2) $g^{-1}f \in K[x_1, \ldots, x_n]$ and $A_n/A_n g^{-1}(d + f)$ is a simple A_n-module, where A_n is the Weyl algebra on $x_1, \ldots, x_n$ and their partial derivatives.

We deal with (2) in Section 3. It follows from (1), (2) and Theorem 2·2 that the module $\mathscr{D}(U)/\mathscr{D}(U)(d + f)$ is simple. Thus $\mathscr{D}(X)/\mathscr{D}(X)(d + f)$ is simple by Theorem 2·1. This strategy will be applied to a smooth quadric surface in Section 4.

3. *Weyl algebras*

Let K be a field of characteristic zero. The ring of differential operators of the affine space $\mathbb{A}^n(K)$ is the Weyl algebra $A_n = A_n(K)$. In this section we construct derivations with respect to which $K[\boldsymbol{x}] = K[x_1, \ldots, x_n]$ is d-simple. These derivations are then used in the construction of a family of non-holonomic simple A_n-modules. Stafford's example in [**Sta**, theorem 1·1] is a member of this family.

Let R be a commutative K-domain and consider the ring $S = R[y_1, \ldots, y_r]$. We will assume that the monomials of S are ordered by the graded lex order. This means that we order the monomials according to their degree; however, if two monomials have the same degree, then the bigger monomial is the one that is higher in the lexicographical order. The leading monomial of a polynomial of S is its monomial of highest graded lex order. A polynomial in S is monic if the coefficient of its leading monomial is 1. The support of a polynomial f of degree k in S is the set of multi-indices α of length k such that y^α has non-zero coefficient in f. We will assume that S is endowed with a derivation d which satisfies:

(1) $d(R) \subseteq R$;

(2) R is d-simple;

(3) there exist $a_i, b_i \in R$ such that $d(y_i) = a_iy_i + b_i$ for $i = 1, \ldots, n$.

Define a map $\lambda\colon \mathbb{N} \to R$ by $\lambda(\alpha) = -\sum_{i=1}^{r} a_i\alpha_i$ where $\alpha = (\alpha_1, \ldots, \alpha_r)$. We wish to show that such a derivation exists when R is a polynomial ring in one variable over K, and that S is d-simple. Recall that the length of a multi-index $\alpha \in \mathbb{N}^r$ is $|\alpha| = \alpha_1 + \ldots + \alpha_r$. We begin with a technical lemma.

Lemma 3·1. *If a d-ideal J of S contains a polynomial h of degree k then it contains a monic polynomial of the same degree whose support is contained in the support of h.*

Proof. If $g \in R$ then the leading term of $d(gy^\alpha)$ is $(d(g) - \lambda(\alpha)g)y^\alpha$. Suppose that the leading monomial of h is gy^α, where α has length k. Write $h_0 = h$ and assume, by induction, that a sequence $h_0, h_1, h_2, \ldots, h_{k-1}$ of elements of J has been found such that h_j has leading term $d^j(g)y^\alpha$ and support contained in the support of h. Write $h_k = d(h_{k-1}) + \lambda(\alpha)h_{k-1}$. Then h_k has leading term $d^k(g)y^\alpha$, unless $d^k(g) = 0$. Since R is d-simple, $d^k(g) = 0$ can only happen if g is invertible. But in this case h_{k-1} is monic, as required. Moreover $h_k \in J$ and its support is contained in the support of h. Thus we can assume that J contains an infinite sequence $h_0, h_1, \ldots$ of elements satisfying the above hypotheses. But since R is d-simple, there exist $q_1, \ldots, q_s \in R$ such that

$$\sum_{j=0}^{s} q_j d^j(g) = 1.$$

Thus $\sum_{j=0}^{s} q_jh_j \in J$ has y^α as its leading monomial and its support is contained in the support of h. This proves the lemma.

Proposition 3·2. *Suppose that $S = R[y_1]$ and that $d(q) \neq a_1q + b_1$ for all $q \in R$. Then S is d-simple.*

Proof. Suppose that S is not d-simple and let J be a proper non-zero d-ideal of S. We will aim at a contradiction. Let f be a non-zero element of S of smallest possible degree k in y_1. By Lemma 3·1 we can assume that f is monic. Then $d(f) - ka_1f \in J$ has degree $\leqslant k - 1$ and so must be zero. Let $c \in R$ be the coefficient of the term of degree $k - 1$ of f. Equating the term of degree $k - 1$ of $d(f) - ka_1f$ to zero we get

$$d(c) + kb_1 - a_1c = 0, \quad \text{hence} \quad d(-c/k) = a_1(-c/k) + b_1,$$

a contradiction. Thus S is d-simple.

This proposition is due to Shamsuddin; see [**Arc**, theorem 2·3·16]. It is the key to the construction of an important family of derivations of $K[\boldsymbol{x}]$ with respect to which this ring is d-simple. Throughout this section the partial differential operator $\partial/\partial x_i$ of $K[\boldsymbol{x}]$ will be denoted by ∂_i.

Theorem 3·3. *Let $a_2, \ldots, a_n, b_2, \ldots, b_n$ be non-zero polynomials in $K[x_1]$. If*

(1) $a_i/a_j \notin \mathbb{Q}$ *whenever* $2 \leqslant i < j \leqslant n$ *and*

(2) $\deg(a_i) > \deg(b_i)$ *for* $i = 2, \ldots, n$;

then $K[x_1, \ldots, x_n]$ is d-simple with respect to the derivation

$$d = \partial_1 + \sum_{i \geqslant 2} (x_ia_i + b_i)\partial_i.$$

Proof. By Proposition 3·2 and induction it is enough to prove that if $1 \leqslant k < n$ then for each $k < i \leqslant n$ and all $f \in K[x_1, \ldots, x_k]$ one has $d(f) \neq fa_i + b_i$. This statement can be proved by induction on k. For $k = 1$ and $0 \neq f \in K[x_1]$, we have that $d(f) = \partial f / \partial x_1$ has degree $< \deg(f)$. On the other hand, $fa_i + b_i$ has degree equal to $\deg(f) + \deg(a_i)$, since $\deg(a_i) > \deg(b_i)$ for all $i \geqslant 2$. Thus $fa_i + b_i$ has degree $> \deg(f)$ and $d(f) \neq fa_i + b_i$, for $i \geqslant 2$.

Suppose, by induction, that the result is true for the polynomial ring in $k - 1$ variables. Let $f \in K[x_1, \ldots, x_k]$ be such that $d(f) = fa_i + b_i$ for some $k < i \leqslant n$. By the induction hypothesis $f \notin K[x_1, \ldots, x_{k-1}]$. Thus we can write $f = \sum_{j=0}^{m} c_j x_k^j$ with $c_m \neq 0$ and $m \geqslant 1$. A calculation shows that

$$d(f) = \sum_{j=0}^{m} (d(c_j) + jc_j a_k + (j+1)c_{j+1} b_k) x_k^j$$

where we are assuming that $c_j = 0$ if $j > m$. Comparing the leading coefficients of $d(f)$ and $fa_i + b_i$ as polynomials in x_k we arrive at the equation

$$d(c_m) = (a_i - ma_k)c_m. \tag{3·4}$$

But d restricts to a derivation of $K[x_1, \ldots, x_{k-1}]$. By Proposition 3·2 and the induction hypothesis $K[x_1, \ldots, x_{k-1}]$ is d-simple. Hence we conclude from (3·4) that c_m is invertible; in particular $c_m \in K$. Since $k < i$, it follows by hypothesis that $a_i - ma_k \neq 0$ and so c_m must be zero, a contradiction. Thus $d(f) \neq fa_i + b_i$ for $k < i$ and all $f \in K[x_1, \ldots, x_k]$, as required.

We will now use these derivations to construct simple A_n-modules. Let d be the derivation of Theorem 3·3. The adjoint action of d on A_n induces a derivation δ in the graded ring $K[\boldsymbol{x}, \Xi] = K[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$ of A_n. If $g \in K[\boldsymbol{x}, \Xi]$ is homogeneous of degree m in the ξs and $\sigma_m(a) = g$ then

$$\delta(g) = \sigma_m([d, a]) = \{\sigma_1(d), g\}.$$

A straightforward calculation shows that:

(1) $\delta(g) = d(g)$ if $g \in K[\boldsymbol{x}]$;

(2) $\delta(\xi_i) = -a_i \xi_i$,

for $i = 2, \ldots, n$. A technical lemma is required before we tackle the main theorem; the above notation remains in force.

LEMMA 3·5. *Assume that $a_2, \ldots, a_n$ are linearly independent over $\mathbb{Q}$. Let J be a δ-ideal of $K[\boldsymbol{x}, \Xi]$ homogeneous with respect to Ξ. If $0 \neq h \in J$ then J contains ξ^α for some multi-index α in the support of h (as a polynomial in the ξs).*

Proof. Among all the elements of J, choose one whose support is as small as possible and is contained in the support of h. Without loss of generality we can assume that this element is h itself. Applying Lemma 3·1 with $R = K[\boldsymbol{x}]$ and $y_i = \xi_i$ we conclude that J contains a monic element whose support is contained in the support of h. Thus, without loss of generality, we can suppose that h is monic. Moreover, since J is homogeneous, we can also suppose that h is homogeneous. Hence

$$h = \xi^\alpha + \sum_{\beta < \alpha} g_\beta \xi^\beta.$$

where $\beta < \alpha$ in the lexicographic order and $g_\beta \in K[\boldsymbol{x}]$. But

$$\delta(h) - \lambda(\alpha)h = \sum_{\beta<\alpha}(d(g_\beta) + (\lambda(\beta) - \lambda(\alpha))g_\beta)\xi^\beta$$

has fewer non-zero summands than h. By the minimality of the support of h all the coefficients of $\delta(h) - \lambda(\alpha)h$ must be zero. Thus

$$d(g_\beta) = (\lambda(\alpha) - \lambda(\beta))g_\beta.$$

Since $K[\boldsymbol{x}]$ is d-simple, it follows that $g_\beta \in K$. But $a_2, \ldots, a_n$ are linearly independent over $\mathbb{Q}$, so $\lambda(\alpha) \neq \lambda(\beta)$ and $g_\beta = 0$ for all $\beta \neq \alpha$. Therefore $h = \xi^\alpha \in J$ and the lemma is proved.

Recall that $\mathscr{C}_k$ stands for the K-vector space of operators of order at most k of A_n. The operators ∂^α with $|\alpha| \leqslant k$ form a basis of $\mathscr{C}_k$.

Theorem 3·6. *For $2 \leqslant i \leqslant n$, let a_i, b_i, h_i be non-zero polynomials in $K[x_1]$ such that*:

(1) *$a_2, \ldots, a_n$ are linearly independent over $\mathbb{Q}$ and*

(2) *$deg\,(a_i) > \max\,\{\deg(b_i), \deg\,(h_i)\}$ for $i = 2, \ldots, n$.*

If

$$d = \partial_1 + \sum_{i\geqslant 2}(x_i a_i + b_i)\partial_i.$$

and $h = \sum_{i\geqslant 2} h_i x_i$ then $A_n/A_n(d+h)$ is a simple non-holonomic A_n-module.

Proof. Suppose that $A_n(d+h)$ is not a maximal left ideal of A_n. Hence there exists a left ideal J that contains $A_n(d+h)$ properly. Let A_{n-1} be the K-subalgebra of A_n generated by x_i and ∂_i, for $i \leqslant n-1$. Note that A_n is generated by $x_1, \ldots, x_n$ and by the derivations d and $\partial_2, \ldots, \partial_n$. Thus any element of A_n is congruent, modulo $A_n(d+h)$, to an element in $A_{n-1}[x_1]$, which we will call its residue. If $D \in \mathscr{C}_k$ then its residue has order at most k. Moreover, if D belongs to J, then so does its residue, since $d + h \in J$.

Let k be the smallest possible order of a non-zero element of $J \bigcap A_{n-1}[x_1]$. We can assume that $k \geqslant 1$; otherwise $J = A_n$. By Lemma 3·5, the ideal J contains an element of the form $\partial^\alpha + P$, where $\alpha \in \mathbb{N}^n$ has length k and first coordinate zero, and $P \in \mathscr{C}_{k-1}$. But the residue of P also belongs to $\mathscr{C}_{k-1}$. Thus replacing P by its residue if necessary, we can assume that $P \in A_{n-1}[x_1] \bigcap \mathscr{C}_{k-1}$. Hence we have shown that J contains an element

$$D = \partial^\alpha + \sum_\beta g_\beta \partial^\beta \in A_{n-1}[x_1]$$

where $|\alpha| = k$ is the smallest possible order for a non-zero element of $A_{n-1}[x_1]$ and $|\beta| \leqslant k-1$.

Let β be an n-tuple of non-negative integers whose first coordinate is zero. Write e_i for the n-tuple which has 1 in the ith coordinate and zeroes elsewhere. We will use the following easily proved identities:

$$[d, \partial^\beta] = \lambda(\beta)\partial^\beta$$

$$[h, \partial^\beta] = -\sum_{i=2}^{n} \beta_i h_i \partial^{\beta - e_i}.$$

It follows from these identities that

$$[d+h, D] - \lambda(\alpha)D \equiv \sum_{i=2}^{n} (d(g_i) + a_i g_i - \alpha_i h_i)\partial^{\alpha - e_i} \pmod{\mathscr{C}_{k-2}},$$

where $g_i = g_{\alpha - e_i}$. But $[d+h, D] - \lambda(\alpha)D \in J \cap A_{n-1}[x_1]$ has order smaller than k, so it is zero. In particular

$$d(g_i) + a_i g_i - \alpha_i h_i = 0 \tag{3·7}$$

for $i = 2, \ldots, n$. Note that this implies that $g_i \neq 0$ for some $i = 2, \ldots, n$. Suppose that $g_{i_0} \neq 0$. Applying ∂_j, $j \neq 1$ to (3.7), we obtain

$$\partial_j(d(g_{i_0})) + a_{i_0}\partial_j(g_{i_0}) = 0.$$

Since $[d, \partial_j] = -a_j \partial_j$ when $j \neq 1$, this last equation becomes

$$d(\partial_j(g_{i_0})) + (a_j + a_{i_0})\partial_j(g_{i_0}) = 0.$$

Since $K[\boldsymbol{x}]$ is d-simple by Theorem 3·3, it follows that $\partial_j(g_{i_0}) \in K$. Moreover, if $\partial_j(g_{i_0}) \neq 0$ then $a_{i_0} + a_j = 0$, which contradicts the hypotheses. Thus $\partial_j(g_{i_0}) = 0$ for $j = 2, \ldots, n$, and so $g_{i_0} \in K[x_1]$. Therefore $d(g_{i_0}) = \partial g_{i_0}/\partial x_1$ and

$$\deg(d(g_{i_0})) < \deg(g_{i_0}) < \deg(a_{i_0} g_{i_0} - \alpha_{i_0} h_{i_0}),$$

a contradiction. Thus the theorem is proved.

Let $\lambda_2, \ldots, \lambda_n \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$, and consider the element

$$\mathbf{s} = \partial_1 + \sum_{i=2}^{n} (\lambda_i x_i x_1 + 1)\partial_i + \sum_{i=2}^{n} 2x_i.$$

Stafford proved in [**Sta**, theorem 1·1] that $A_n/A_n\mathbf{s}$ is a simple A_n-module, thus showing that there exist non-holonomic simple A_n-modules when $n \geqslant 2$; see also [**KrL**, theorem 8·7]. Of course this example is a special case of Theorem 3·6. Note that we have turned Stafford's example from a right module into a left module with the help of the standard transposition; see [**Cou**, chapter 16, section 2] for details. This particular example does not work when the field of constants is $\mathbb{Q}$. However, Theorem 3·6 can be applied in this case too; it is enough to choose $a_i = x_1^i$ and $b_i = h_i = 1$, for example. Finally, in [**Sta**, proposition 2·2], Stafford gives another example of a non-holonomic simple module, this time over A_2, which cannot be written in the form of Theorem 3·6. It would be interesting to see if the theorem can be extended to include this second example.

4. *Quadric surfaces*

We shall now combine the results of Sections 2 and 3 to construct simple $\mathscr{D}(X)$-modules when X is a non-singular quadric surface. Let K be an algebraically closed field of characteristic zero. The equation $x_1^2 + x_2 x_3 = 1$ determines a non-singular quadric X in $\mathbb{A}^2(K)$.

Let U be the principal open set of X defined by $x_1 \neq 0$. The coordinate ring $\mathscr{O}(U)$ is obtained by localising $\mathscr{O}(X)$ at the powers of x_1. Now U is a select variety and its module of derivations is free with basis

$$\Delta_2 = \partial_2 - x_3(2x_1)^{-1}\partial_1 \quad \text{and} \quad \Delta_3 = \partial_3 - x_2(2x_1)^{-1}\partial_1.$$

By Theorems 3·3 and 2·2 the ring $\mathscr{O}(U)$ is δ-simple with respect to the derivation $\delta = \Delta_2 - (1 - x_2x_3)\Delta_3$. Moreover

$$d = -2x_1\delta = (x_3 - x_2x_1^2)\partial_1 - 2x_1\partial_2 + 2x_1^3\partial_3$$

is a derivation of $\mathscr{O}(X)$. The following result comes from [**Arc**, theorem 2·5·23], but we give a more direct proof.

Theorem 4·1. *$\mathscr{O}(X)$ is d-simple with respect to* $d = (x_3 - x_2x_1^2)\partial_1 - 2x_1\partial_2 + 2x_1^3\partial_3$.

Proof. Let J be a non-zero d-ideal of $\mathscr{O}(X)$. Since the minimal primes over a d-ideal are also d-ideals, we can assume that J is prime. Let J_{x_1} be the localization of J at the powers of x_1. Thus J_{x_1} is a d-ideal of $\mathscr{O}(U)$. But $d = -2x_1\delta$, so $\mathscr{O}(U)$ is a d-simple ring. Hence $1 \in J_{x_1}$. Consequently J contains a power of x_1 and, because it is prime, it contains x_1. Since J is a d-ideal we get that $d(x_1) = x_3 - x_2x_1^2 \in J$. Therefore $x_3 \in J$. But this implies that $1 = x_1^2 + x_2x_3 \in J$, which completes the proof.

It does not seem to be known whether the ring of coordinates of a non-singular quadric hypersurface is d-simple. Unfortunately the obvious generalization of the derivation of Theorem 4·1 to higher dimension does not work.

Theorem 4·2. *Let* $h = 2x_1x_3$ *and* $d = (x_3 - x_2x_1^2)\partial_1 - 2x_1\partial_2 + 2x_1^3\partial_3$. *Then* $\mathscr{D}(X)/\mathscr{D}(X)(d + h)$ *is a non-holonomic simple left* $\mathscr{D}(X)$*-module.*

Proof. Put $M = \mathscr{D}(X)/\mathscr{D}(X)(d + h)$. Localizing at x_1 we have that

$$M_{x_1} \cong \mathscr{D}(U)/\mathscr{D}(U)(\delta - x_3).$$

Now $\mathscr{D}(U)$ contains the Weyl algebra A_2 generated by x_2, x_3 and Δ_2, Δ_3. Moreover $\delta - x_3 \in A_2$. By Theorem 3·6 the module $A_2/A_2(\delta - x_3)$ is simple. Thus by Theorem 2·2(2) it follows that M_{x_1} is a simple $\mathscr{D}(U)$-module. Finally, by Theorem 2·1, we conclude that M itself is a simple $\mathscr{D}(X)$-module.

This approach is not limited to quadric surfaces; it can also be applied to any select variety. The proof of the next result is omitted since it is a straightforward application of Theorems 2·2 and 3·6.

Theorem 4·3. *Let X be a select variety of $\mathbb{A}^m(K)$. Suppose that X has dimension $n \leqslant m$ and coordinates $x_1, \ldots, x_n$. For $2 \leqslant i \leqslant n$, let a_i, b_i, h_j be non-zero elements of $K[x_1]$ such that*:

(1) *$a_2, \ldots, a_n$ are linearly independent over $\mathbb{Q}$ and*

(2) $\deg(a_i) > \max\{\deg(b_i), \deg(h_i)\}$ *for $i = 2, \ldots, n$;*

where deg *denotes the degree as a polynomial in x_1. If $d = \partial_1 + \sum_{i\geqslant 2}(x_ia_i + b_i)\partial_i$ and $h = \sum_{i\geqslant 2} h_ix_i$ then $\mathscr{D}(X)/\mathscr{D}(X)(d + h)$ is a simple $\mathscr{D}(X)$-module.*

Here is a very easy recipe for constructing simple examples of select varieties. Let $F(u_2, \ldots, u_n)$ be a polynomial of degree $\geqslant 1$ and coefficients in K; and let $k \geqslant 1$ be an integer. Denote by X the variety of $\mathbb{A}^{n+1}(K)$ defined by

$$u_1^k + F(u_2, \ldots, u_n) = 0 \quad \text{and} \quad u_1v = 1.$$

Proposition 4·4. *X is a select variety of dimension $n - 1$.*

Proof. Let I be the ideal of X in the polynomial ring $K[u_1, \ldots, u_n, v]$. Put $y = v+I$ and $x_i = u_i + I$, for $1 \leqslant i \leqslant n$. By [**McR**, proposition 15·1·17], the module $\Omega^1(X)$ is generated by $dx_1, \ldots, dx_n$ and dy, subject to the relations

$$kx_1^{k-1}dx_1 + \sum_{i=2}^{n}(\partial F/\partial x_i)dx_i \quad \text{and} \quad ydx_1 + x_1 dy = 0.$$

From these relations we deduce that

$$dx_1 = -k^{-1}y^{k-1}\sum_{i=2}^{n}(\partial F/\partial x_i)dx_i \quad \text{and} \quad dy = -y^2 dx_1.$$

Thus $\Omega^1(X)$ is a free module with basis $\{dx_2, \ldots, dx_n\}$. Hence X is a select variety with coordinates $x_2, \ldots, x_n$.

Of course one would like to have more examples of varieties to which these techniques could be applied; especially of varieties which were not select.

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