ON INVARIANT LINE ARRANGEMENTS

R. DE MOURA CANAAN AND S. C. COUTINHO

ABSTRACT. We classify all arrangements of lines that are invariant under foliations of degree 4 of the real projective plane.

1. INTRODUCTION

The history of the problem that we study here can be traced to a paper [5] published by Gaston Darboux in 1878. Darboux's paper is an exposition of a new method to compute holomorphic first integrals of polynomial differential equations over the complex projective plane. By that, Darboux means an equation defined by a 1-form

$$\Omega = Adx + Bdy + Cdy$$

where A, B and C are nonconstant homogeneous polynomials of the same (positive) degree that satisfy $xA+yB+zC = 0$. A 1-form whose coefficients are homogeneous of the same degree defines a field of planes in $\mathbb{C}^3$. The relation $xA + yB + zC = 0$ is required for this field of planes to induce a field of lines in $\mathbb{P}^2(\mathbb{C})$. To solve the differential equation $\Omega = 0$ by Darboux's method one computes a first integral using the algebraic curves of $\mathbb{P}^2(\mathbb{C})$ that are invariant under Ω. In modern parlance, an algebraic curve C of $\mathbb{P}^2(\mathbb{C})$, defined as the zero locus of a nonconstant homogeneous polynomial $F \in \mathbb{C}[x, y, z]$, is *invariant* under Ω if $\Omega \wedge dF = F\eta$ for some 2-form η with polynomial coefficients. For the connection between systems of ordinary differential equations on the plane and Darboux's differential equations over $\mathbb{P}^2(\mathbb{C})$ see [8, Chapter V, p. 469].

An important example discussed by Darboux concerns lines that are invariant under a vector field. His starting point are the equations studied by Jacobi in [9]. Defining the degree d of Ω to be

$$d = \deg(A) - 1 = \deg(B) - 1 = \deg(C) - 1,$$

Jacobi's equations correspond to the case $d = 1$. Although these equations always have invariant lines, this is not true for general systems of degree greater than 1. Indeed, as Jouanolou showed in [10, p. 157ff], a generic Ω of degree greater than one does not have any invariant algebraic curves. Therefore, the minimum number of invariant lines that a projective equation of degree $d \geq 2$ can have is zero. On the other hand, given an integer $d > 2$, it is always possible to write a 1-form Ω of degree d with infinitely many invariant algebraic lines. For example, all lines through $[0 : 0 : 1]$ are invariant under $\Omega = h(ydx - xdy)$, where h is a homogeneous

Date: July 27, 2012.

1991 *Mathematics Subject Classification.* Primary: 52C30, 34C05; Secondary: 14N20.

Key words and phrases. derivation, singularity, invariant line, arrangement.

The first author was supported by a scholarship from CNPq(PIBIC) and the second author was partially supported by grants from CNPq and Pronex/Faperj.

polynomial of degree d. Since such a form must have a rational first integral, one may ask whether an upper bound exists on the maximal number of invariant lines of projective 1-forms of degree d, *which do not have a rational first integral*. If, like Darboux, we consider all the lines in $\mathbb{P}^2(\mathbb{C})$ defined by homogeneous linear polynomials with *complex coefficients*, then the answer is $3d$; see [3, p. 110].

A related trend emerged in the study of Hilbert's 16th problem. Researchers working in that field made a conjecture about the maximal number $\alpha(d)$ of *real* lines that are solutions of a system of polynomial differential equations whose degree is at most d. In the language we have been using such a system translates into a projective differential equation of degree d with *real* coefficients, under which the line at infinity is invariant. The conjecture stated that

$$\alpha(d) = \begin{cases} 2d+2 & \text{if } d \text{ is even} \\ 2d+3 & \text{if } d \text{ is odd,} \end{cases}$$

and it was shown to be correct for $d = 2, 3$ and 4; see [13] and [12]. However, it turned out to be false for all the other degrees for which it was checked, see [1].

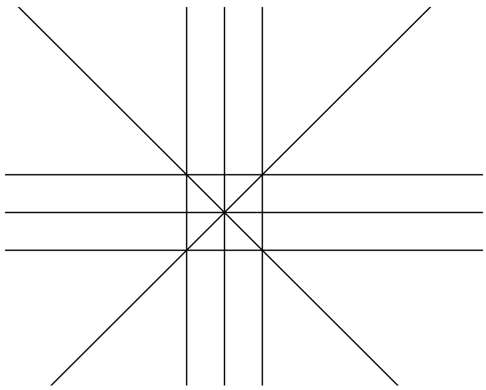

FIGURE 1. The affine Darboux arrangement of order 3 with 8 lines

Actually, the approach used in [1] to prove the conjecture false consisted in first abstracting the properties that an arrangement of lines in the real projective plane must satisfy in order to be invariant under a differential equation of degree d. These *projective Darboux arrangements of order d*, as we call them in this paper, are formally defined in section 2. In [1] the authors used Grümbaum's catalogue of simplicial arrangements (see [7]) to select those that are Darboux and then used a computer to determine a differential equation with those invariant lines. This allowed them to find lower bounds for $\alpha(d)$ that settled the conjecture in the negative for $5 \leq d \leq 20$. However, except for those degrees for which the conjecture holds, we only know $\alpha(d)$ for $d = 5$; indeed $\alpha(5) = 15$, by [1, Theorem 7, p. 210]. At the end of their paper Artés, Grünbaum and Llibre conjectured that every real Darboux arrangement of order d has $2d + O(1)$ lines. This conjecture is one of the research problemas mentioned in [2, Conjecture 5, p. 313].

In the more recent [11], Llibre and Vulpe classified the arrangements of 9 lines (counted with multiplicity and including the complex lines and the line at infinity) that are invariant under a system of real differential equations of degree 3. In particular, they show that, up to isomorphism, there is only one maximal Darboux arrangement of order 3 invariant under a differential equation of degree 3: the one whose affine lines are shown in Figure 1.

This paper began as an attempt to do, for the arrangements invariant under differential equations of degree four, what [11] had done for degree three. However, our approach is quite different from the one followed by Llibre and Vulpe. To begin with we classify the 10-line (projective) Darboux arrangements of order four from a purely geometrical point of view, with no reference to the differential equations, which are only determined *a posteriori*. This allows us to answer in the *negative* a question that comes up naturally when the arrangements are determined independently of the equations; namely,

> is every 10-line (projective) Darboux arrangement of order k invariant under a system of differential equations of degree k?

Also unlike [11], we use the computer algebra system AXIOM [4] in order to simplify the case-by-case analysis that our approach requires. However, while Llibre and Vulpe showed that, up to isomorphism, there is only one 9-line arrangement invariant under a differential equation of degree 3, we found so many that we had to group them into families, some of which are infinite.

The paper is organised as follows. In section 2 we give a formal definition of Darboux arrangements, both projective and affine, and prove some of their elementary properties. The section ends with a detailed description of the strategy that we use to classify Darboux arrangements of order four. The classification itself is dealt with in sections 3 and 4. The former contains the classification of those arrangements that have lines in at least four different directions, while all the remaining arrangements are classified in the latter section. Finally, in section 5, we return to the problem that motivated this paper and determine which Darboux arrangements of order four are invariant under a real polynomial differential equation of degree four.

2. Preliminaries

We begin with a precise definition of the kind of arrangement of lines that will be considered throughout this paper. As explained in the introduction, the properties these arrangements are required to satisfy are common to all arrangements invariant under a projective differential equation in Darboux's sense; see [1, Proposition 6, p. 209] for the details.

2.1. **Darboux arrangements.** Let $\mathcal{P}$ be an arrangement of lines in the real projective plane. As usual, the points of intersection of any two of its lines will be called *vertices*. We say that $\mathcal{P}$ is a *projective Darboux arrangement of order* n if it satisfies the following properties:

(a) no line in the arrangement contains more than $n+1$ vertices;
(b) at most $n+1$ lines of the arrangement go through the same vertex;
(c) the arrangement has $O(n^2)$ vertices.

As shown in [12], a projective Darboux arrangement of order four that is invariant under a differential equation cannot have more than 10 lines. Our aim in this paper is to go one step further, by

- classifying the Darboux arrangements of order 4, up to (affine) isomorphism, and
- determining which ones are invariant under a real polynomial differential equation of degree four.

We begin by using projective transformations to position the arrangement in a way that simplifies the analysis. This part of the argument holds for arrangements of all orders, so we state it in full generality. Applying to $\mathcal{P}$ a projective transformation, if necessary, we may assume that one of the lines of this arrangement is given in homogeneous coordinates by $z = 0$. Dehomogenizing the other lines of $\mathcal{P}$ with respect to z we obtain an arrangement $\mathcal{A}$ of lines in the affine plane with the following properties:

(1) no line in the arrangement has more than n vertices;
(2) no more than n lines of the arrangement can be parallel;
(3) at most $n + 1$ lines of the arrangement go through the same vertex;
(4) the arrangement has $O(n^2)$ vertices.

Note that (1) and (2) follow from property (a) of the projective Darboux arrangements since, by hypotheses, the line at infinity of $\mathcal{A}$ belongs to $\mathcal{P}$. An *affine Darboux arrangement of order* n will be an arrangement of lines in the affine plane obtained by dehomogenizing with respect to z a projective Darboux arrangement of order n one of whose lines is $z = 0$.

Lemma 2.1. *An arrangement of the affine plane that satisfies conditions (1) and (4) above for some* $n \geq 1$ *and which contains*

- *at least a pair of parallel lines, and*
- *at least two lines in different directions,*

also satisfies conditions (2) and (3).

Proof. Suppose that $\mathcal{A}$ is an arrangement that satifies the hypotheses of the lemma. If $\mathcal{A}$ contains m parallel lines in a given direction, these lines intersect any line not parallel to them at m distinct points. Since $\mathcal{A}$ must contain such a nonparallel line by hypotheses, we have that $m \leq n$; which gives condition (2). Since $\mathcal{A}$ contains at least a pair of parallel lines, the proof of (3) follows from a similar argument with the lines through a point playing the rôle previously played by the parallel lines. □

As one easily checks, the lines

$$z, y - x, x - kz, y - kz \quad \text{for} \quad 1 \leq k \leq n \tag{2.1}$$

define a *projective* Darboux arrangement of order n. Here, and throughout the paper, we identify a line with the polynomial whose vanishing defines it. Since the above arrangement has $2n + 2$ lines, it follows that any *maximal affine* Darboux arrangement of order n has at least $2n + 1$ lines. That is why we will assume, from now on, that $\mathcal{A}$ is an *affine Darboux arrangement of order* n with at least $2n + 1$ lines. These arrangements satisfy the following properties.

Proposition 2.2. *Let* $\mathcal{A}$ *be an affine Darboux arrangement of order* $n \geq 4$ *with at least* $2n + 1$ *lines. Then:*

(1) $\mathcal{A}$ *contains lines in at most* $n + 1$ *different directions;*
(2) $\mathcal{A}$ *contains at least two pairs of parallel lines in two different directions.*

Lines	*Equations*	
ℓ	$(1-b)y - b(c-1)x - (1-b)b$	$by - c(1-b)x - b^2$
ℓ'	$(1-b)y - b(c-1)x - (1-b)c$	$by - c(1-b)x - bc$
λ	$y + b(x-1)$	$y - (1-b)x - b$

TABLE 1. The lines of Lemma 2.3

Proof. Let $\mathcal{P}$ be the projective arrangement whose dehomogenization is $\mathcal{A}$. Since z belongs to $\mathcal{P}$, intersecting the other lines of $\mathcal{P}$ with z gives rise to at most $n+1$ distinct vertices. But these vertices correspond to the directions that the lines of $\mathcal{A}$ can take, which proves (1). Moreover, if $\mathcal{A}$ has at least $2n+1$ lines in at most $n+1$ directions, we must have parallel lines among those of $\mathcal{A}$. On the other hand, there cannot be more than n parallels in any given direction, which implies that there are parallel lines in at least two distinct directions, proving (2). □

Let $\mathcal{A}$ be an affine Darboux arrangement of order n with at least $2n+1$ lines. Applying a translation followed by a transvection to the lines of $\mathcal{A}$ and changing the scale in the coordinate axis, we can assume that the two pairs of parallels predicted in Proposition 2.2 are given by the polynomials $x, x-1, y, y-1$. The arrangement formed by these four lines will be denoted by $\mathcal{Q}$, and called the *standard square*. Since we are interested in arrangements up to affine isomorphism, the argument above shows that we may assume, without loss of generality, that $\mathcal{A}$ contains $\mathcal{Q}$. Under this hypothesis, the four lines of $\mathcal{Q}$ will be called the *standard lines* of $\mathcal{A}$ and the four points at which they intersect its *standard vertices*.

Let $\mathcal{A}$ be an affine Darboux arrangement. Since we are assuming that $\mathcal{A}$ contains $\mathcal{Q}$, any affine isomorphism σ that we apply to $\mathcal{A}$ must satisfy $\mathcal{Q} \subset \sigma(\mathcal{A})$. We say that such a σ is a *Darboux isomorphism* of $\mathcal{A}$. Note that this is not equivalent to saying that σ stabilizes $\mathcal{Q}$, because a different set of four lines may take over the rôle of standard square configuration in the image of $\mathcal{A}$ under σ.

2.2. **Darboux arrangements of order four.** Using the terminology of the previous article, we can state the aim of this paper as follows:

> Classify, up to Darboux isomorphism, all 9-line affine Darboux arrangements of order four.

Our first result is concerned with certain affine Darboux arrangements of order four with only seven lines. Here, as indeed throughout the whole paper we will assume that all Darboux arrangements contain the standard square.

Lemma 2.3. *Suppose that, besides the standard square $\mathcal{Q}$, a Darboux arrangement of order four contains two distinct parallel lines ℓ and ℓ' and a seventh line λ, in a total of four different directions. If neither ℓ nor ℓ' contains a standard vertex then, up to Darboux isomorphism, the lines ℓ, ℓ' and λ are those of table 1, where $b \neq c$ are real numbers both different from 0 and 1.*

Proof. We argue by contradiction. Both ℓ and ℓ' must cross every one of the four lines of the standard square. The only way they would not contain four different intersection points is if two of those points coincided. But this would mean that ℓ goes through one of the standard vertices, a contradiction. Therefore ℓ and ℓ' contain 4 *distinct* vertices each. Since the arrangement has order four, λ must cross

ℓ and ℓ' at one of these vertices. Applying a Darboux isomorphism we can assume that λ crosses ℓ at the point where it intersects $x = 0$. This implies that if the equations of ℓ and ℓ' are, respectively, $y = ax + b$ and $y = ax + c$ then the equation of λ can be written in the form $y = \alpha x + b$. Note that $a, b, c \neq 0, 1$ because, by hypothesis, these lines do not contain any standard vertex and are not parallel to any of the lines of $\mathcal{Q}$. Now $y = 0$ intersects $x = 0$, $x = 1$, ℓ and ℓ' at the points with x-coordinates 0, 1, $-b/a$ and $-c/a$. Since λ contains $(0, b)$ and is different from x and ℓ, it follows that it cannot contain $(0, 0)$ and $(-b/a, 0)$. Hence, λ must intersect $y = 0$ at $(1, 0)$ or $(-c/a, 0)$. We analyse these two cases separately. Note that the hypotheses on ℓ and ℓ' imply that $b \neq c$ are both nonzero.

Suppose, first, that this intersection occurs at $(1, 0)$. In this case λ has equation $y = -b(x - 1)$, so its intersection with $y = 1$ occurs at a point whose abscissae is $(b-1)/b$. However, this point must coincide with one of the four vertices already on $y = 1$, whose abscissae are 0, 1, $(1-b)/a$ and $(1-c)/a$. Equating $(b-1)/b$ with each one of these numbers we find that b must be equal to 1, $-a$, or $a(b - 1) = (1 - c)b$. The first case is excluded by hypothesis, and the second implies that λ is parallel to ℓ, so it too is not allowed. On the other hand, if we multiply the equations of ℓ and ℓ' by $b - 1$ and use $a(b - 1) = (1 - c)b$ we end up with the equations given in the first column of table 1.

Assume now that λ intersects $y = 0$ at $(-c/a, 0)$. In this case the equation of λ is $cy - abx - bc$ which intersects $y = 1$ at a point whose abscissae is $c(1 - b)/ab$. Arguing as in the previous case and discarding those possibilities that contradict the hypotheses, we find that $c(1 - b) = ab$. Taking this into account the equations for ℓ, ℓ' and λ can be rewritten as in the second column of table 1. □

2.3. **An overview of the argument.** We end this section with an overview of the strategy used in the case-by-case analysis required to identify the arrangements. Since we have often to deal with several parallel lines, it is convenient to say that two parallel lines are *twins*, three are *triplets*, and so on. In section 3 we classify the maximal Darboux arrangements of order four with lines in at least four different directions which do not have any quadruplets. More precisely, after showing that in a maximal Darboux arrangement of order four with two pairs of non-standard parallel lines, at least one line of each pair must contain a standard vertex, we consider arrangements with

- a non-standard twin pair with *exactly* one line through a standard vertex and another with *at least* one line through such a vertex (Proposition 3.3);
- both lines of each non-standard twin pair go through exactly one standard vertex (Proposition 3.5);
- both lines of each non-standard twin pair go through at least one standard vertex, and at least one of them goes through two such vertices (Proposition 3.1).

These arrangements are listed in table 6. More precisely, this table contains two pairs of parallel lines for each of the four families of Darboux arrangements classified according to the strategy described above, and an additional column with the possible ninth lines for each one of these families.

Since a Darboux arrangement of order four without a triplet must fall into one of the families listed above, we assume in section 4 that all arrangements have at least three parallel lines. Actually, these arrangements must have either a quadruplet or

Type	First twin pair	Second twin pair	Choice of ninth lines	
1	$y-x$	$y+x-1$	$-x+2y$	
	$y-x+1$	$y+x$	$y+1$	
2	$-x+y$	$\zeta x+y-1$	$(\zeta-1)x+y-1$	$-(\zeta+1)x+y+\zeta$
	$-x+y+\zeta$	$\zeta x+y-\zeta^2$	$(\zeta-1)x+y$	
3	$y-ax$	$y-a(x-1)-1$	$-x+y$	
	$y-a_1x$	$y-a_1(x-1)-1$		
4	$y-x$	$y-a(x-1)-1$	$x+y-1$	
	$-x+ya$	$-x+ya-a+1$		

TABLE 2. The maximal Darboux arrangements with four different directions, note that $\zeta=\varphi, 1-\varphi$ and $a\neq\pm1$.

two triplets, because those arrangements with only one triplet where already found in section 3. Taking this into account we consider arrangements with

- one quadruplet and one triplet (article 4.2);
- two triplets but no quadruplet (article 4.3);
- one quadruplet and no triplet (article 4.4).

The Darboux arrangements with triplets are listed in table 7, and the ones with a quadruplet but no triplets in table 8.

3. Four distinct directions

Let $k\geq 8$ be an integer. Throughout this section $\mathcal{A}$ denotes a k-line Darboux arrangement of order four with four pairs of parallel lines, each one of which in a different direction.

3.1. **The strategy.** In order to classify these arrangements we do a careful case-by-case analysis. Denoting by E_1 and E_2 the two non-standard pairs of twins, the cases we consider can be formulated as follows:

case 1: no lines of E_1 contain a standard vertex;
case 2: only one line of E_1 contains a standard vertex;
case 3: both lines of E_1 contain standard vertices.

There are a few things that we should point out. First, in both cases 1 and 2, we are not, in principle, imposing any restrictions on E_2. However, given *case 1*, it follows that in *case 2* we may assume that at least one line of E_2 contains a standard vertex. Similarly, given *cases 1 and 2* we may assume that each line of both E_1 and E_2 contains a standard vertex. Thus, if V is the set of standard vertices,

$$E_1=\{\ell_1,\ell_2\} \text{ and } E_2=\{\lambda_1,\lambda_2\},$$

then we can rewrite the above cases as follows:

case 1: $\ell_1\cap V=\ell_2\cap V=\emptyset$;
case 2: $\ell_2\cap V=\emptyset$ while $\ell_1\cap V\neq\emptyset\neq\lambda_1\cap V$;
case 3: $\ell_j\cap V\neq\emptyset\neq\lambda_j\cap V$, for $j=1,2$.

We include a detailed proof of a subcase of *case 3* as a sample of the kind of argument that we use to prove the results of this section.

As will become apparent in this proof, the analysis tends to produce many arrangements that turn out to be Darboux isomorphic. Checking whether two given arrangements are isomorphic or not can be very time consuming. We approach this problem in two stages. In the first stage the arrangements are sorted by the degree vectors of their lines. More precisely, given an arrangement A and a vertex v of A, the *degree* of v is the number of lines through v, which is equal to the degree of the vertex in the graph of A. The *degree vector* of a line $\ell \in A$ is the vector whose entries are the degrees of the vertices of A that belong to ℓ. We always enumerate these degrees in increasing order. To each arrangement $\Delta(A)$ we associate the multiset whose elements are the degree vectors of the lines of A. Using the notation of the computer algebra system AXIOM, we represent these multisets as vectors preceded by a multiplicity that indicates the number of times that vector appears in the multiset. For instance, for the arrangement T_1 illustrated in figure 2,

$$\Delta(T_1) = \{2 : [2, 2, 2, 4], 4 : [2, 3, 4], [2, 2, 3, 3], [4, 4]\}$$

Of course two arrangements with different Δ-multisets cannot be isomorphic. Although the converse is not true (see Remark 3.4), these multisets are very helpful in checking whether two given arrangements are Darboux isomorphic, as we explain in the proof of Proposition 3.1.

As will become clear later on in this section, we automated the part of the argument that consists of case checking. However, although a completely automatic classification of the arrangements is possible, it turned out to be unsuitable. Indeed, given a very general input, the algorithms return dozens of arrangements, many of which are Darboux isomorphic. Since checking whether two arrangements are isomorphic can be a rather subtle matter, it proved better to process the arrangements only after the number of parameters had been reduced by the addition of some extra conditions that the lines are required to satisfy.

3.2. **A special case.** In this article we give a complete proof that only one Darboux arrangement exists that satisfies a set of very simmetrical conditions.

Proposition 3.1. *Suppose that $\mathcal{A}$ is a Darboux arrangement of order four with eight lines, such that the two lines of both non-standard twin pairs of $\mathcal{A}$ pass through standard vertices. If one of the non-standard lines contains two standard vertices then, up to affine isomorphism,*

$$\mathcal{A} = \{x, y, x - 1, y - 1, y - x, y + x - 1, -x + 2y, y - x + 1\}.$$

Proof. Up to Darboux isomorphism we may assume that the line ℓ that contains two standard vertices is $y = x$ and that its twin ℓ' is $y = x - 1$. Denoting by λ and λ' the lines of the second twin pair, we know from the hypotheses that λ' is (1) parallel to λ and (2) contains a standard vertex. Assume, further, that

(3) λ' does not contain $(0, -1)$, $(1, 0)$ or $(2, 1)$, which are the three vertices already in ℓ'.

Under these hypotheses, λ must intersect ℓ' in one of the vertices listed in (3). Since $\lambda \neq \ell'$ must also contain a standard vertex, it follows that λ is equal to

$$2y - x, \quad y - (2x - 1), \qquad \text{or} \quad y - a(x - 1). \tag{3.1}$$

We analyse each case separately.

(iii) $y = ax + 1$;

Since $y = -x + 1$ contains $(1, 0)$, it follows that $a = -1$ is only allowed for the lines (i) and (ii), giving the 8-line Darboux arrangements A_4 and A_5, respectively.

Assuming, from now on, that $a \neq -1$, let us consider λ' to be $y = ax$. In this case $y = 1$ intersects the other lines of the arrangement in the points whose abscissae are 0, 1, 2, $(a + 1)/a$ and $1/a$. Thus the arrangement will be Darboux only if two of these numbers coincide. Since $a \neq 0, 1, -1$ it follows, by equating the abscissae, that the only viable case is $a = 1/2$, which gives the arrangement A_1. A similar argument can be used when λ' is $y = a(x - 1) + 1$. The points of intersection of $x = 0$ with the other lines of the arrangement have ordinates equal to -1, 0, 1, $-a$ and $-a + 1$. Dismissing those values of a that are not allowed by hypothesis, we find that two of these numbers coincide only if $a = 2$, which corresponds to the arrangement A_3.

Taking now λ' to be $y = ax + 1$ and intersecting it with the other lines, we find the points $(1, a + 1)$, $(0, 1)$, as well as

$$\left(\frac{-1}{a}, 0\right), \quad \left(\frac{-1}{a-1}, \frac{-1}{a-1}\right), \text{ and } \left(\frac{-2}{a-1}, \frac{-(a+1)}{a-1}\right).$$

Two of these five points coincide only if $a = -1, 0, 1$. Since all these values have already been excluded, we have no Darboux arrangements in this case. This settles the case for which either λ and λ' cross ℓ' at a new vertex.

When both λ and λ' cut ℓ' at one of the vertices already in ℓ', we have three possibilities for either of these lines; namely, those listed in (3.1), where a is a nonzero real number. But λ and λ' must be parallel. Therefore, either $a = 1/2$ and the arrangement is A_1, or $a = 2$, and the arrangement is A_3.

In order to finish the proof we must show that all these six arrangements are Darboux isomorphic. To begin with, they have the same Δ-multiset, namely

$$\{2 : [2, 2, 2, 4], 4 : [2, 3, 4], [2, 2, 3, 3], [4, 4]\}.$$

Therefore, all these arrangements have exactly one line with only two vertices, each of degree four. It turns out that in A_4 this line is $y = 0$, the origin is one of the points of degree four and the vertices $(0, 1)$ and $(1, 0)$ have degrees three and four, respectively. Thus, to an arrangement A_i, with $i \neq 4$, we apply an affine isomorphism that moves the $[4, 4]$ line to the x-axis and and places one of its vertices at the origin. To the arrangement that results from this transformation we apply an affine isomorphism that moves to the y-axis one of the non-horizontal lines through the origin and places one of its vertices of degree 3 at the point $(0, 1)$. This is repeated for each line and point of degree three until the resulting arrangement is found to be isomorphic to A_3. Note that if this never happened we would have proved that the A_i is not Darboux isomorphic to A_4. However, it turns out that all the arrangements of table 3 are Darboux isomorphic. Table 4 gives the affine isomorphisms that transforms the arrangements A_i into A_4.

□

3.3. **The main algorithms.** Since computers are far better at enumeration then humans, we defer to them the case-by-case analysis required in most of the other proofs. The main algorithm we use, called `makeDarboux`, takes as inputs

- an arrangement $\mathcal{A}$ of lines, some of which depend on various parameters,

Arrangement	Affine isomorphism
A_1 (minus sign)	$(x+y,x)$
A_1 (plus sign)	$(x+y,y)$
A_2	$(-x+1,-x-y+1)$
A_3	$(-y+1,-x-y+1)$
A_5	$(-y+1,x)$

TABLE 4. The affine isomorphisms of Proposition 3.1.

- a *teste list* T, which contains equations describing those values the parameters are not allowed to take;
- the polynomial condition $\mathfrak{g}$ that the parameters must satisfy, and
- the number m of distinct lines the arrangement is expected to have.

The last number is included because some choices of parameters may force two lines to coincide, thus reducing the actual number of lines obtained at the end of the computation.

The algorithm uses two lists `Retry` and `OutCase`, each element of which is a `Record`, a data type defined in the AXIOM manual as an object "composed of one or more other objects, each of which is referenced with a selector"; see [4, section 2.4, p. 164]. The selectors we use in the `Records` mentioned above are called `arrangement` and `condition`. The latter points to a list which contains the polynomial conditions that must be satisfied by the parameters of the lines listed under `arrangement`. The list `Retry` is initialized with $[\mathcal{A},\mathfrak{g}]$, which we use as a shorthand for the record

$$[\text{arrangement} = \mathcal{A}, \text{condition} = \mathfrak{g}],$$

while `OutCase` is left empty. `Retry` is the list of the arrangements still to be analysed, so the algorithm stops when it is empty.

Assuming that `Retry` is non-empty, we pick its first element, say $[L,g]$. Recall that the elements of g (the conditions) are polynomials in the parameters. The algorithm runs through the conditions one by one, searching for those that are linear in one of the parameters. Once such a condition is found, the algorithm solves it for the linear parameter. Taking the resulting expressions into L and g, we obtain a new arrangement and a new list of conditions, that we continue to denote by the same letters. Having eliminated all conditions with a linear parameter, the algorithm checks whether L is Darboux when its parameters satisfies g. This is done using a function called `darboux?`, which is discussed below. If `darboux?` returns `true` and $\#L = m$, the `Record` $[L,g]$ is added to `OutCase` and the algorithm chooses the next element of `Retry`. However, when g consists of one equation in only one parameter, $[L,g]$ is added to `OutCase` only if this equation has a real root. Let us consider now how the algorithm proceeds when $[L,g]$ is not Darboux.

Given a line $\ell \in L$, let P_ℓ be the set of points where ℓ intersects the other lines of L. When L is not Darboux, the algorithm `makeDarboux` constructs the list $\mathcal{B}$ whose elements are the lines $\ell \in L$ for which $\#P_\ell > 4$. Note that such lines must exist because we are assuming that it has already been determined that L is not Darboux. In order to make of L a Darboux arrangement, at least two of the points of P_ℓ must be made to coincide for each $\ell \in \mathcal{B}$. To this end the program uses a Gröbner factorization algorithm (see [6, p. 264]) to create a new list C whose elements are the conditions, together with g, that make two points coincide in P_ℓ, for some $\ell \in \mathcal{B}$. Since these conditions are Gröbner bases, they are actually sets of polynomials in the parameters of L. Each condition is then separately checked. Let c be a condition in C that does not belong to the teste list T. If `darboux?` returns `true` when applied to $[L, c]$ and $\#L = m$, then this `Record` is added to `OutCase`, otherwise it is put into of `Retry`, for further analysis.

A `Record`, in the output of `makeDarboux`, that corresponds to a Darboux arrangement with one equation in one parameter can be further processed using `solveForParameter`. This function solves the equation that represents the condition and substitutes the resulting roots into the arrangement. In order to have exact results we use the `Axiom` function `radicalSolve`, which returns the roots as surds. This is possible because all the parameters we have to deal with are roots of quadratic equations. Another closely related function is `findNinth`, which uses `makeDarboux` to search for a ninth line that, when added to a given 8-line arrangement, turns it into a 9-line Darboux arrangement.

Two of the functions implemented as part of `makeDarboux` will be used separately. The first, called `intersectAll`, takes as inputs an arrangement $\mathcal{A}$, a line $\ell \in \mathcal{A}$ and the conditions that its parameters must satisfy, and returns the points of intersection of ℓ with each one of the lines of $\mathcal{A}$. The second, called `makeCoincide`, takes a set of points P, depending on parameters, and returns all the conditions on the parameters that will make two points of P coincide. Besides the arrangement, `makeCoincide` also takes as inputs a list of equations that correspond to values the parameters cannot take (a *test list*) and the polynomial condition the parameters must satisfy. A typical input of `makeCoincide` is the output of `intersectAll`.

The function `darboux?`, mentioned above, returns true or false according to whether conditions (1) and (4) of the definition of affine Darboux arrangements are satisfied or not; see Lemma 2.1. Its inputs are an arrangement and the polynomial condition its parameters must satisfy. Using `intersectAll` the function determines the vertices that belong to each line and checks that there are no more than four per line and $4^2 = 16$ in total. Note that `darboux?` and most of the other functions that we have implemented assume that the arrangement being tested has order four.

3.4. **Eight line arrangements with four directions.** To make the procedure more systematic we subdivide the analysis in a few propositions, each one of which corresponds to a less generic case than the previous one.

Proposition 3.2. *In every maximal Darboux arrangement of order four with two (nonparallel) non-standard twin pairs, at least one line of each pair must contain a standard vertex.*

Proof. Let (ℓ, ℓ') and (λ, λ') be the twin pairs of non-standard lines and assume, by contradiction, that neither ℓ nor ℓ' contains a standard vertex. Then, ℓ, ℓ' and λ can be written in the form given in Lemma 2.3; see Table 2. We analyse the two possibilities separately.

In the first case, λ is the line $y = -b(x-1)$, so the equation of λ' can be written in the form $y = -bx+\gamma$, for some real number γ. In the second case, λ has equation $y = (1-b)x+b$, so that λ' is the line $y = (1-b)x+\gamma$, for some $\gamma \in \mathbb{R}$. The remainder of the analysis was automated using the algorithm `makeDarboux` described in the article 3.3. In the first case we obtain a Darboux arrangement with eight lines, whose non-standard twin pairs are

$$b^2x+y-b, b^2x+y-b^2+b-1, bx+y-b, bx+y-1$$

but which is not contained in any 9-line Darboux arrangement of order four unless b is chosen so that at least one of the lines above goes through a standard vertex. In the second case, there is not even an eighth line arrangement with the required properties. □

It follows from Proposition 3.2 that we need analyse only arrangements with both pairs of parallels having a line through a standard vertex, which correspond to cases 2 and 3 of §3.1. Denoting by φ the *golden section* $(\sqrt{5}+1)/2$, let $\zeta \in \{\varphi, 1-\varphi\}$. From now on we use "$A$ is an arrangement of type n" as a shorthand for "A is *Darboux isomorphic* to an arrangement of type n".

Proposition 3.3. *Suppose that one of the non-standard twin pairs of an* 8*-line Darboux arrangement* $\mathcal{A}$ *has* exactly *one line through a standard vertex and the other* at least *one line through such a vertex. Then, up to affine isomorphism, the arrangement must be of type 2.*

Proof. Let ℓ, ℓ' and λ, λ' be the two non-standard pairs of parallels of $\mathcal{A}$. Applying an isometry, if necessary, we may assume that

- ℓ has equation $y = ax$;
- ℓ' has equation $y = ax + b$ with $b \neq 0, 1$ and $a + b \neq 0, 1$.

In particular, ℓ' must cross the standard lines in the four *distinct* points

$x=0$	$x=1$	$y=0$	$y=1$
$(0,b)$	$(1,a+b)$	$(-b/a,0)$	$((1-b)/a,1)$

Moreover, we can also assume that λ goes through either $(0,0)$, $(0,1)$ or $(1,1)$, for we may use an isometry to swap $(1,0)$ with $(0,1)$ without affecting the other hypotheses. Thus, λ must go through one of these three points and one of the four points in ℓ'. The possible equations are given in the table below, where the choices that give rise to one of the standard lines are marked with $\times$.

Points	$(0,b)$	$(1,a+b)$	$(-b/a,0)$	$(1-b/a,1)$
$(0,0)$	$\times$	$y=(a+b)x$	$\times$	$y=\frac{a}{1-b}x$
$(0,1)$	$\times$	$y=(a+b-1)x+1$	$y=\frac{ax+b}{b}$	$\times$
$(1,1)$	$y=(1-b)x+b$	$\times$	$y-1=\frac{a(x-1)}{a+b}$	$\times$

TABLE 5. The possible λs through a vertex of ℓ' and $(0,0)$, $(0,1)$ or $(1,1)$.

Now λ' must be parallel to one of the lines in the table above, so it can be written as the sum of the homogeneous linear component of λ with a constant c. The possible values of c, listed in Table 6, have been determined by taking into account that λ' must also go through one of the four vertices already present in ℓ'.

For each λ in Table 5, we construct a parallel line λ' using the data from Table 6. Applying `makeDarboux` with inputs

λ	Possible values for c			
$ax+by-y$	$-a\,b-b^2+b$	b	$-b^2+b$	$\times$
$ax+bx-y$	$(ab+b^2-b)/a$	b	$(ab+b^2)/a$	$\times$
$xa+xb-x-y+1$	$(ab+b^2-2b+1)/a$	1	$(ab+b^2-b)/a$	b
$-xa+ya+yb-b$	$-a^2-2ab+a-b^2$	$-b$	$-a-2b+1$	$-ab-b^2$
$-xa+yb-b$	$-ab+a-b^2$	$-b$	$-2b+1$	$-b^2$
$xb-x+y-b$	$(-a+b^2-2b+1)/a$	$-b$	$-a-2b+1$	$(b^2-b)/a$

TABLE 6. The possible values of c in Proposition 3.3

Arrangement: $[x, y, x-1, y-1, y-ax, y-ax-b, \lambda, \lambda']$;
Condition: 0;
Test list: $[0, 1, a, b, b-1, a+b-1, a+b]$;
Number of lines: 8

we end up with 38 Darboux arrangements of order four with eight lines each. The functions `isoType1?` and `isoType2?` can then be used to show that 22 of these arrangements are of Type 1 and must be discarded, the other 16 are isomorphic to the 8-line arrangement L made up of four twin pairs of Type 2. Searching for a ninth line compatible with L, we find that that there are three, each one giving rise to a non-isomorphic arrangement of Type 2. □

Remark 3.4. *Actually, Type 2 is the only one for which an* 8*-line arrangement gives rise to three non-isomorphic* 9*-line Darboux arrangements. Moreover, two of these* 9*-line arrangements, namely those whose ninth lines are* $(\zeta-1)x+y-1$ *and* $(\zeta-1)x+y$*, have the same* Δ*-set; see Figure 3.*

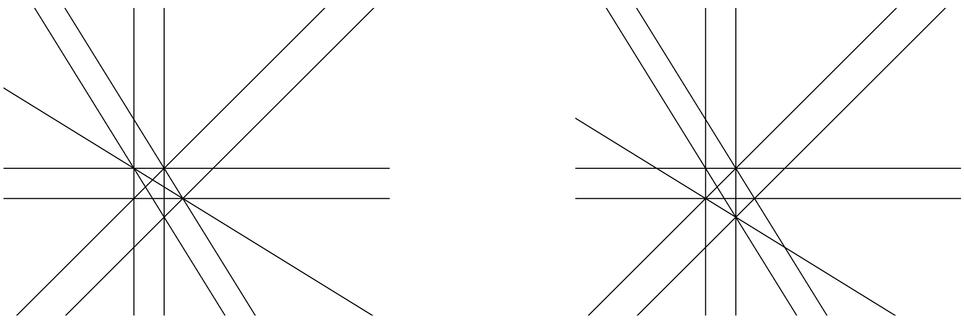

FIGURE 3. Non-isomorphic arrangements of Type 2 with the same Δ-set

It follows from Propositions 3.2 and 3.3 that the only possible Darboux arrangements with 4 twin pairs of lines, each in a different direction are those whose nonstandard twin pairs have both lines through a standard vertex. However, some of these arrangements have already been classified in Proposition 3.1; namely, those for which at least one line in a nonstandard twin pair contains two standard vertices. Thus we need only deal with eight line arrangements all of whose nonstandard twins go, each one, through exactly one standard vertex.

Proposition 3.5. *Suppose that both lines of each non-standard twin pair go through exactly one standard vertex. The corresponding Darboux arrangements are affinely isomorphic to the arrangements of types 3 or 4.*

Proof. Let ℓ, ℓ' and λ, λ' be the two nonstandard twin pairs. Suppose first that one line from each non-standard twin pair goes through the same standard vertex. Without loss of generality we may assume this vertex to be $(0,0)$ and that $\ell = y-ax$ and $\lambda = y-a_1x$. Since, by hypothesis, these lines cannot contain a second standard vertex, we must have that $a \neq a_1$ are both different from 0 and 1. On the other hand ℓ' must contain a standard vertex and be parallel to ℓ. Since an analogous condition must hold for λ', we can assume that $\ell' \in \{y-a(x-1),\, y-a(x-1)-1\}$ and $\lambda' \in \{y-a_1x-1,\, y-a_1(x-1)-1,\, y-a_1(x-1)\}$. Choosing $\ell' = y-a(x-1)-1$ and $\lambda' = y-a_1(x-1)-1$ we obtain an 8-line Darboux arrangement with no need for any further restrictions on a and a_1. Applying `findNInth` to this arrangement, we end up with 4 distinct arrangements, one of which is of type 3 (with no other restriction on a and a_1), the other three of type 4 (for which $a = a_1$). Of these three the simplest was chosen to represent the type 4 arrangements, in order to prove that the other two are isomorphic to it we must apply a change of variables followed by change of parameter. Take, for example, the arrangement whose nonstandard lines are

$$xa_1+ya_1-2y-a_1+1,\ -xa_1+y+a_1-1,\ x+ya_1-2y-a_1+1,\ -xa_1+y,\ x+ya_1-2y.$$

Applying the Darboux isomorphism

$$D(x,y) = \left[x, \frac{-x+(a_1-1)y}{a_1-2}\right]$$

we end with an arrangement whose nonstandard lines are

$$x+y-1,\ x(a_1-1)-y-a_1+2,\ x(a_1-1)-y,\ x-y(a_1-1)+a_1-2, x-y(a_1-1)$$

which is seen to be of type 4 by taking $a = a-1$. The other arrangement may be dealt with in a similar way. Thus, up to isometry, may can assume from now on that $\ell' = y-a(x-1)$. However, applying `makeDarboux` and eliminating those arrangements with lines through two standard vertices we get four 8-line arrangements, all of them easily shown to be isomorphic to Type 2.

Suppose now that there are no two lines, among the non-standard four, that go through the same point. Up to affine isomorphism, the possible pairs are either

$$[y-ax,\ y-a(x-1)] \text{ or } [y-ax, y-a(x-1)-1].$$

In each case, the hypotheses we are assuming uniquely determine the second pair, so the two non-standard pairs will be either

$$[y-ax, y-a(x-1), y-a_1x-1, y-a_1(x-1)-1] \text{ or}$$
$$[y-ax, y-a(x-1)-1, y-a_1x-1, y-a_1(x-1)].$$

To the arrangements obtained by adding to $\mathfrak{Q}$ the above nonstandard twin pairs we apply the function `makeDarboux`, followed by `findNinth`. However, all the Darboux arrangements found by these algorithms either have a quadruple, in which case they will be described in article 4.2, or are isomorphic to one that has a line through two standard vertices (Type 1) and must be discarded. □

Type	Parameter values		Complementary lines	
	a	b		
5	a	a	$x - c$	$y - c$
6	a	a	$y - a^2$	$y - ax$
7	2	2	$y - 3$	$-x + y - 1$
8	2	2	$y - x - 1$	$y - x + 1$
9	$1 - b$	b	$y - x - b$	$y - x - b + 1$
10	$\zeta + 1$	$-\zeta$	$y - x + \zeta$	$x + \zeta y$
11	$1/b$	$b \neq \pm 1$	$-bx + y$	$-xb^2 + y$
12	a	$b \neq a$	$-bx + ay$	$(1 - b)x + (a - 1)y - a + b$
13	$-\zeta$	$1 - \zeta$	$(\zeta - 2)x + y$	$(1 - \zeta)x + y - (1 - \zeta)$

TABLE 7. 9-line Darboux arrangements that contain $\mathcal{B}$, recall that $\zeta = \varphi, 1 - \varphi$ and that $a, b \neq 0, 1$.

4. One quadruplet or two triplets

In this section we study Darboux arrangements of order four, with nine lines when there are either a quadruplet or two triplets. The complete list is given in tables 7 and 8. As usual all arrangements are assumed to contain the set $\mathcal{Q}$ of standard lines.

4.1. **Triplets.** Suppose that the Darboux arrangement $\mathcal{A}$ has triples in two different directions. Performing an affine transformation, if necessary, we can assume that $\mathcal{A}$ contains the subarrangement

$$\mathcal{B}' = \{x,\, x - 1,\, x - a,\, y,\, y - 1,\, y - b\},$$

where a and b are assumed to be different.

Proposition 4.1. *Let $\mathcal{A}$ be a Darboux arrangement that contains $\mathcal{B}'$ as a subarrangement. Any line of $\mathcal{A}$ that is not parallel to a line of $\mathcal{B}'$, must pass through two of the nine vertices of $\mathcal{B}'$.*

Proof. Let ℓ be a line of $\mathcal{A}$ that is not parallel to any line of $\mathcal{B}'$. Intersecting ℓ with the lines of $\mathcal{B}$ we get 6 points, which cannot all be distinct. Indeed, ℓ cannot contain more than four vertices of $\mathcal{A}$. But this is possible only if the intersections of ℓ with two of the horizontal lines each coincides with the intersection of ℓ with some vertical line, which is equivalent to the statement of the proposition. □

Note that an arrangement of the type we are considering must contain a line that is neither horizontal nor vertical, otherwise we would have more than 4 parallel lines in one direction, contradicting the definition of Darboux arrangement. So we may also assume, from now on, that $\mathcal{A}$ contains a line transversal to those of $\mathcal{B}'$. Performing a translation, followed by a change of scale of the axis and a reflection, if necessary, we can take the transversal line to be $y = x$. Thus we may assume that we are dealing with a Darboux arrangement $\mathcal{A}$ that contains

$$\mathcal{B} = \mathcal{B}' \cup \{y - x\} = \{x,\, x - 1,\, x - a,\, y,\, y - 1,\, y - b,\, y - x\},$$

as a subarrangement. In particular, $\mathcal{A}$ contains the vertices,

$$V = \{(0,0),(0,1),(0,b),(1,0),(1,1),(1,b),(a,0),(a,1),(a,b)\}.$$

4.2. **One quadruplet and one triplet.** Suppose that $\mathcal{A}$ contains at least one quadruplet, which must necessarily be either horizontal or vertical. Without loss of generality we will assume that the extra line is horizontal and equal to $y = c$. Thus, $y - x$ contains the vertices

$$(0,0),(1,1),(a,a),(b,b),(c,c),$$

which is only possible if two of these coincide, say $a = c$. Therefore, $\mathcal{A}$ contains the eight lines

$$x,\ x-1,\ x-a,\ y,\ y-1,\ y-a,\ y-b,\ y-x,$$

and there is only one more line ℓ that we need to add to make $\mathcal{A}$ into a 9-line Darboux arrangement. Now, although ℓ cannot be horizontal, it can be vertical. But in this case, an argument analogous to the one above shows that $\ell = x - b$, and we get an arrangement of type 5.

Thus, we may assume that ℓ is neither horizontal nor vertical. Since $y - x$ intersects the four horizontal lines at distinct points, it follows that either ℓ goes through one of these points or it is parallel to $y - x$. If ℓ and $y - x$ intersect then they can do so only at one of the four vertices of $y - x$, whose coordinates are $(0,0)$, $(1,1)$, (a,a) and (b,b). The first three coincide with points where a vertical line of the arrangement intersects a horizontal line also in the arrangement. This allows us to translate the corresponding point to the origin, so that $\ell = y - \alpha x$ for some nonzero real number α. Using `makeDarboux`, we find three arrangements of type 6. The point (b,b) has to be handled separately because it is not contained in any vertical line of the arrangement. However, using `makeDarboux` we find that no 9-line Darboux arrangement contains a line of the form $y - \alpha(x-b) - b$. Assuming now that ℓ and $y - x$ are parallel, we have that $\ell = y - x - \beta$, for some nonzero real number β. By Proposition 4.1 this line must go through two vertices in V. Applying `makeDarboux`, we find six arrangements, all of which turn out to be of type 7.

4.3. **Two triplets but no quadruplets.** Assume now that $\mathcal{A}$ has two triplets none of which is a quadruplet. Suppose, first, that ℓ is parallel to $y - x$. Then it has the form $y - x - \beta$. But, as in the previous article, ℓ must contain two among the nine points of V, so that β can only take the values 1, b or $b - a$. Applying `makeDarboux` followed by `findNinth`, to the arrangements obtained by adding ℓ to $\mathcal{B}$, we end up with three arrangements, all of them isomorphic to type 8, when $\beta = 1$, and four arrangements of type 8, one of type 9 and four of type 10, when $\beta = b$; see table 7. No Darboux arrangement was found when $\beta = b - a$.

Next we consider the arrangements for which ℓ and $y-x$ intersect. Arguing as in article 4.2, we can assume that ℓ is of the form $y-\alpha x$ or $y-\alpha x+a(\alpha-1)$. Applying `makeDarboux` followed by `findNinth` to the arrangement whose eighth line is $y-\alpha x$ we obtain two arrangements of types 11, one of type 12 and four of type 13. Doing the same for the arrangement with eighth line equal to $y - \alpha x + a(\alpha - 1)$ we end up with an arrangement of type 11 and four of type 13.

Type	Parameter values		Complementary lines	
	a	b		
14	$1/3$	$1/2$	$-x-y+1$	$x-2y$
15	$-1/(b-2)$	b	$(b-1)x-y+1$	$-bx+y$
16	a	$a^2/(2a-1)$	$(1-a)x-y+a$	$ax-y$
17	a	b	$bx-y$	$ax-y$
18	$3/2$	2	$x-y+1$	$x+y-2$
19	-1	-2	$x-y-1$	$x-y-2$
20	$b-1$	ζ	$x-y+(b-1)$	$bx+y-b$
21	$b-1$	b	$x-y+(b-1)$	$y-bx$

TABLE 8. Maximal Darboux arrangements with one quadruplet and no other triplets, recall that $\zeta = \varphi, 1-\varphi$ and that $a, b \neq 0, 1$.

4.4. **One quadruplet and no other triplet.** This is the only case in this section that does not fall into the remit of Proposition 4.1. However, after an affine transformation we can assume that any maximal Darboux arrangement $\mathcal{A}$ with one quadruplet but no other triplet contains the lines

$$x, x-1, y, y-1, y-a, y-b, \tag{4.1}$$

with $a \neq b$ and both different from 0 and 1. Since each of the vertical lines already contains four vertices, all the other lines of $\mathcal{A}$ pass through two of these vertices. We can assume, up to Darboux isomorphism, that $y-x$ is a line of $\mathcal{A}$. Any other line $\ell \in \mathcal{A}$ is either parallel or incident to $y-x$. In the former case we can take $\ell = y-x-\beta$ where $\beta \in \{1, a\}$; while in the latter we can take ℓ to be $y-ax$, $y+x-1$, $y+(1-a)x-1$, $y+ax-a$, $y+(a-1)x-a$ or $y+(a-b)x-a$. Therefore,

$$\mathcal{B}_\ell = \{x, x-1, y, y-1, y-a, y-b, y-x, \ell\} \subset \mathcal{A},$$

for the possible choices of ℓ listed above. Following our practice in the previous articles, we apply `makeDarboux`, followed by `findNinth`, to $\mathcal{B}_\ell$ for each possible choice of ℓ. The only value of ℓ from which no Darboux arrangement is derived is $\ell = y+(a-b)x-a$.

In all the other articles we were able to sort the Darboux arrangements generated by the program according to the non-isomorphic types in a more or less ad-hoc way. However, the steps described above generate so many different examples of arrangements that we had to develop a function specially for the sake of classifying them according to the types listed in table 8. This is made easier for the arrangements considered in this article because they have only one quadruplet, made up of four horizontal lines. For every pair of parallel lines transveral to the quadruplet, the program `classifyArt44` computes the four vertices along each of these lines and runs through all triples of these vertices searching for the one that, once moved to the points $[0,0]$, $[1,0]$ and $[0,1]$, makes the resultant arrangement isomorphic to one of those listed in table 8.

However, there are some arrangements that `classifyArt44` is unable to identify correctly. Some of these are of Type 6 (see table 7) and are easily detected by comparing Δ-sets. The remaining are obtained by concatenating

$$[x, x-1, y, y-1, y-x]$$

with one of the following subarrangements

$$\begin{aligned}
&[y-a, y-b, \quad xa - x + y - a, \quad xb - x + y - b]\\
&[-xb + x + yb - b,\ -2xb + x + yb,\ y - b,\ yb - 2b + 1]\\
&[-xb + yb - y + b,\ -2xb + x + yb - y + b,\ y - b,\ yb - y + b]\\
&[-x + y - b + 1,\ y - b,\ y - b + 1,\ x(b-2) + y - b + 1].
\end{aligned}$$

These arrangements require both a change of variables and a change of parameters before they can be made to look like one of the types in table 8. For example, the change of variables

$$x = -x + 1 \text{ and } \ y = -y + 1$$

transforms the arrangement determined by the first pair of lines into one whose nonstandard lines are

$$-y - a + 1, -y - b + 1, x - y, -xa + x - y, -xb + x - y, \tag{4.2}$$

which is of type 17, as one sees by taking $a = 1 - a$ and $b = 1 - b$ in (4.2). A similar argument shows that the other two pairs of lines give rise to arrangements of type 15.

5. Differential equations

We must now return to the problem we started from, which consisted in determining those arrangements of lines that are invariant under a real polynomial differential equation of degree four. Following [1] we wrote an Axiom function, based on the method of undetermined coefficients, to find the required differential equations. Recall that Llibre and Vulpe proved in [11] that, up to affine isomorphism, there exists only one real polynomial differential equation of degree three with an 8-line invariant Darboux arrangement, the one whose lines are

$$x, x-1, x-2, y, y-1, y-2, y-x, y+x. \tag{5.1}$$

After grouping the Darboux arrangements in families we ended up with 21 non-isomorphic families, some of which are infinite. Moreover, some Darboux arrangements turned out to correspond to no differential equation.

Theorem 5.1. *The Darboux arrangements of type 2 whose ninth lines are* $(\zeta - 1)x + y$ *or* $(\zeta - 1)x + y - 1$ *and the arrangements of types 4, 10, 12 and 13 are not invariant under any real differential equation of degree four.*

This result was proved computationally. The arrangements listed in the theorem are those for which the method of undetermined coefficients returned only zero. However, we should point out that, to speed up the calculations, whenever parameters were present in the arrangements, their values were assumed to be generic. In other words, the Gröbner bases used to solve the polynomial systems from which we computed the differential equations were calculated over the *field of rational functions on the required parameters*. In particular, there may be values of a for which the corresponding arrangements of type 4 are invariant under some differential equation of degree 4. As a consequence of Theorem 5.1 we have that the concept of Darboux arrangement, as defined in this paper, does not characterize

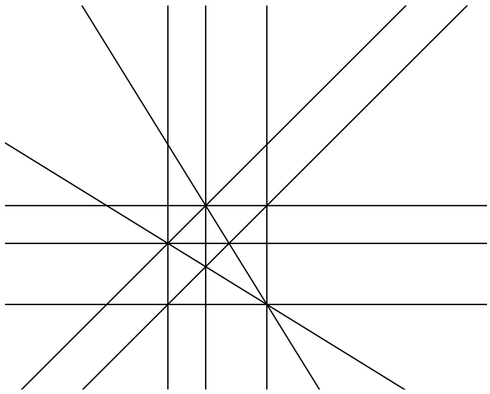

Figure 4. A Darboux arrangement of order four with ten lines

those arrangements that truly interest us; namely, those that are invariant under real polynomial differential equations.

Actually, this is not the only reason why the definition of Darboux arrangement given here proves to be unsatisfactory. It turned out that there are Darboux arrangements of order four with 10 lines, like the one in Figure 4. Of course such an arrangement cannot be invariant under any differential equation of degree four. Not surprisingly, the analysis of the arrangements with two triplets to which the one in Figure 4 belongs is one of the few places in [12] where Sokulski makes direct use of the differential equation, instead of relying on an argument based on the geometric properties of the arrangement. With all that in mind one is led to pose the following problem.

Problem 5.2. *Is there a purely geometric characterisation of the maximal arrangements of real lines that are invariant under a real polynomial differential equation?*

Since the known maximal Darboux arrangements of orders 3 and 5 both attain the maximal number of vertices allowed for their respective orders, we looked at the total number of vertices of each one of the arrangements. It turns out that the only affine Darboux arrangements of order four, with the maximum number of vertices (which is 16), are those of type 12. In particular, the exact number of vertices cannot be used as a criterion to distinguish between maximal Darboux arrangements that are invariant under a differential equation and those that are not. Finally, all the programms used in producing the results of this paper are available in the file `darbouxArrangements` that can be downloaded from

`http://www.dcc.ufrj.br/~collier/folia`.

References

1. Joan C. Artés, Branko Grünbaum, and Jaume Llibre, *On the number of invariant straight lines for polynomial differential systems*, Pacific J. Math. **184** (1998), no. 2, 207–230. MR 1628583 (99f:34035)
2. Peter Brass, William Moser, and János Pach, *Research problems in discrete geometry*, Springer, New York, 2005.
3. Maurício Corrêa, Jr. and Márcio G. Soares, *Counting hypersurfaces invariant by one-dimensional complex foliations*, Real and complex singularities, London Math. Soc. Lecture Note Ser., vol. 380, Cambridge Univ. Press, Cambridge, 2010, pp. 104–113. MR 2759081

4. T. Daly, *Axiom: the thirty year horizon, volume 1: tutorial*, Lulu Press, 2005.
5. G. Darboux, *Mémoire sur les équations différentielles algébriques du* I^{o} *ordre et du premier degré*, Bull. des Sc. Math. (Mélanges) (1878), 60–96, 123–144, 151–200.
6. G.-M. Greuel and G. Pfister, *A singular introduction to commutative algebra*, Springer, Berlin-Heidelberg, 2002.
7. Branko Grünbaum, *A catalogue of simplicial arrangements in the real projective plane*, Ars Math. Contemp. **2** (2009), no. 1, 1–25. MR 2485643 (2010d:51033)
8. Yulij Ilyashenko and Sergei Yakovenko, *Lectures on analytic differential equations*, Graduate Studies in Mathematics, vol. 86, American Mathematical Society, Providence, RI, 2008. MR 2363178 (2009b:34001)
9. C. Jacobi, *De integratione aequationes differentiallis* $(a + a'x + a''y)(xdy - ydx) - (b + b'x + b''y)dy + (c + c'x + c''y)dx) = 0$, J. für die reine und angewandte Mathematik **24** (1842), 1–4.
10. J. P. Jouanolou, *Equations de Pfaff algébriques*, Lect. Notes in Math., vol. 708, Springer-Verlag, Heidelberg, 1979.
11. Jaume Llibre and Nicolae Vulpe, *Planar cubic polynomial differential systems with the maximum number of invariant straight lines*, Rocky Mountain J. Math. **36** (2006), no. 4, 1301–1373. MR 2274897 (2008m:34088)
12. Jacek Sokulski, *On the number of invariant lines for polynomial vector fields*, Nonlinearity **9** (1996), no. 2, 479–485. MR 1384487 (96m:34081)
13. Xi Kang Zhang, *The number of integral lines of polynomial systems of degree three and four*, Proceedings of the Conference on Qualitative Theory of ODE (Chinese) (Nanjing, 1993), no. suppl., 1993, pp. 209–212. MR 1259799 (95a:34047)

Departamento de Ciência da Computação, Instituto de Matemática, Universidade Federal do Rio de Janeiro, P.O. Box 68530, 21945-970 Rio de Janeiro, RJ, Brazil.

E-mail address: rocanaan@gmail.com

E-mail address: collier@impa.br

URL: http://www.dcc.ufrj.br/~collier