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Modules of Codimension One over Weyl Algebras

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1. INTRODUCTION

The representation theory of D-modules over an algebraic variety has been mostly concerned with a special class of modules, the holonomic modules. A D-module is holonomic if it has maximal Gelfand–Kirillov codimension, which is equal to the dimension of the base variety. The interest in holonomic D-modules has two sources: their ubiquity and the fact that their theory is extremely elegant.

However, if the base variety is not a curve, the D-module that corresponds to a single differential equation is not holonomic. Thus the case of one differential equation has not been in the main stream of the theory of algebraic D-modules. In fact, until about 1983 it was widely believed that all irreducible modules over the Weyl algebra were holonomic. The first example of a non-holonomic D-module was given by Stafford [8]. In 1988 J. Bernstein and V. Lunts [1, 7] showed that, in fact, most modules over the Weyl algebra that correspond to a single differential equation are irreducible. Their idea was to use the geometry of the characteristic variety to construct families of irreducible modules of Gelfand–Kirillov codimension one.

Let $\mathscr{M}(X)$ be the category of all modules over the Weyl algebra with characteristic variety X. It follows from the results of Bernstein and Lunts that for a generic hypersurface X, the category $\mathscr{M}(X)$ shares some of the nice properties of the category of holonomic modules. Besides, since X is a hypersurface, all the objects of $\mathscr{M}(X)$ have Gelfand–Kirillov codimension one. In this paper we start a more detailed study of the structure of the category $\mathscr{M}(X)$, for a generic hypersurface X. Section 2 collects the notation and basic facts used throughout the paper. In Section 3, we prove that $\mathscr{M}(X)$ has infinitely many non-isomorphic irreducible modules of

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every possible multiplicity. In Section 4 we show that $\mathrm{Ext}^1(M, N)$ may be an infinite dimensional vector space when M, N are objects in $\mathscr{M}(X)$. This should be compared with the holonomic case: $\mathrm{Ext}^i(M, N)$ is always finite dimensional when M, N are holonomic [2, 2.7.15 and 1.6.6]. Finally, these results are used to construct families of projective non-cyclic ideals for the Weyl algebra in Section 5. Indeed, we show that for most choices of elements a, d in the Weyl algebra there exists a left ideal $I(a, d)$ which is projective and non-cyclic.

2. BASIC RESULTS

The nth complex Weyl algebra A_n is the ring of differential operators of affine space $\mathbb{C}^n$. Its generators will be denoted by $x_1, \ldots, x_n$ and $\partial_1, \ldots, \partial_n$, where ∂_i is the differential operator $\partial/\partial x_i$. From now on we shall fix the integer $n \geq 2$, and write A instead of A_n.

The Bernstein filtration of A is defined by giving degree one to each of the above generators. The graded ring of A with respect to the Bernstein filtration will be denoted by S. We shall write $A(k)$ for the kth step in this filtration and $S(k) = A(k)/A(k-1)$ for the kth homogeneous component of the graded ring S. The *order* $\mathrm{ord}(d)$ of an operator d in A is its degree with respect to the Bernstein filtration. The *symbol map* of order k is the canonical projection σ_k: $A(k) \to A(k)/A(k-1) = S(k)$. Let $y_i = \sigma_1(x_i)$ and $y_{i+n} = \sigma_1(\partial_i)$, for $i = 1, 2, \ldots, n$. Then S is a polynomial ring on the variables $y_1, \ldots, y_{2n}$ over $\mathbb{C}$.

The space $\mathbb{C}^{2n}$ is a symplectic manifold with respect to the standard 2-form $\omega = \sum_{i=1}^n dy_i \wedge dy_{i+n}$. To a function $f \in S$ we shall associate the Hamiltonian vector field h_f defined by

$$\sum_{i=1}^{n} (\partial f/\partial y_{i+n})\partial_{y_i} - (\partial f/\partial y_i)\partial_{y_{i+n}}.$$

The *Poisson bracket* of two functions $f, g \in S$ is defined as $\{f, g\} = h_f(g)$. We may use the Poisson bracket to shadow the non-commutativity of A within S. Let $d, d' \in A$ be two operators of order m and k, respectively, then $\sigma_{m+k-2}([d, d']) = \{\sigma_m(d), \sigma_k(d')\}$. This is a very useful statement, as we will see. An ideal J of S is said to be *involutive* if $\{J, J\} \subseteq J$. In this case we also say that the variety $Z(J) \subseteq \mathbb{C}^{2n}$ is involutive.

Let M be a finitely generated left A-module. Let F be a good filtration for M with respect to the Bernstein filtration. Thus $\mathrm{gr}^F M$ is a finitely generated S-module. Such filtrations always exist because M is finitely generated. The *characteristic ideal* $I(M)$ of M is the radical of the annihilator of $\mathrm{gr}^F M$ in S. It is independent of the choice of the good filtration F

of M used to calculate it. The *characteristic variety* $\mathrm{Ch}(M)$ is the variety of $I(M)$ in $\mathbb{C}^{2n}$. Notice that $I(M)$ is a homogeneous ideal of S; in this case we also say that $\mathrm{Ch}(M)$ is a homogeneous variety of $\mathbb{C}^{2n}$. Here is a simple, but very important, example. Let L be a left ideal of A and put $M = A/L$. Then $I(M)$ is the radical of $\sigma(L)$, where $\sigma(L) = \sum_0^\infty \sigma_k(L \cap A(k))$.

The characteristic ideal of a finitely generated left A-module M is an involutive ideal of S. This important result has been proved by Gabber in [4], using purely algebraic methods. It has many useful applications. For example, it implies that the dimension of $\mathrm{Ch}(M)$ cannot be less than n. This is particularly important since $\dim \mathrm{Ch}(M)$ coincides with the Gelfand–Kirillov dimension of M. The involutivity of the characteristic variety is also the key to the constructions of Bernstein and Lunts that we now discuss.

A homogeneous involutive variety X of $\mathbb{C}^{2n}$ is said to be *minimal* if it does not contain any proper homogeneous involutive subvariety. Bernstein and Lunts showed that if d is an operator of order k in A such that $Z(\sigma_k(d))$ is minimal, then A/Ad is irreducible [1, Theorem A′]. They also proved that most operators in A satisfy this property. To make this more precise, we say that a property $\mathbb{P}$ holds for a *generic* f in $S(k)$ if the set $\{g \in S(k)$: $\mathbb{P}$ does not hold for $g\}$ is contained in a countable union of hypersurfaces in $S(k)$. An often used, even if imprecise, shorthand for this is to say that if $f \in S(k)$ is generic, then $\mathbb{P}$ holds. Now if $k \geq 4$ and f is generic in $S(k)$, then $Z(f)$ is a minimal homogeneous involutive variety. This was proved for $n = 2$ by Bernstein and Lunts in [1] and later generalized to all $n \geq 2$ by Lunts in [7]. In fact, the result follows from Theorem 1 of [7], which is stated below. First a definition. Let $f \in S(k)$; the Hamiltonian vector field h_f *preserves* a subvariety $X \subseteq \mathbb{C}^{2n}$ if it is tangent to X at every smooth point of X.

THEOREM 2.1. *Let $k \geq 4$ be an integer and let f be a generic element of $S(k)$. If X is a homogeneous subvariety of $Z(f)$ preserved by h_f then* $\dim(X) \leq 1$.

The following consequence of Theorem 2.1 is also central to the results in this paper.

LEMMA 2.2. *Let $k \geq 4$, $m \geq 1$, and $P \in S(k)$ be a generic polynomial. If $Q \in S(m)$ satisfies $\{Q, P\} \subseteq S \cdot P + S \cdot Q$, then Q is a multiple of P.*

Proof. Let $J = S \cdot P + S \cdot Q$. Then the hypothesis implies that $\{J, P\} \subseteq J$. Denote by h the hamiltonian operator h_P. Then h is a derivation in S. We show that $h(\mathrm{rad}(J)) \subseteq \mathrm{rad}(J)$. If $f \in \mathrm{rad}(J)$, then there exists a positive integer p such that $f^p \in J$. Since $h(J) \subseteq J$, it follows that $h^p(f^p) \in J$. A straightforward calculation shows that $p!(h(f))^p \in \mathrm{rad}(J)$. Thus $h(f) \in \mathrm{rad}(J)$. We conclude that $h(\mathrm{rad}(J)) \subseteq \mathrm{rad}(J)$, as required. But this implies that the subvariety $Y = Z(J) \subseteq Z(P)$ is preserved by h. Since P is generic, we conclude that $\dim Y \leq 1$. But height$(J) \leq 2$, hence $\dim(Y) \geq 2n - 2 \geq 2$, whenever $n \geq 2$; contradicting Theorem 2.1.

We also use the multiplicity of an A-module. Let M be a finitely generated left A-module with a good filtration F and P a prime ideal of S. The *multiplicity* $m_P(M)$ *of* M *with respect to* P is the length of the S_P-module $(\mathrm{gr}^F M)_P$. The multiplicity thus defined is independent of the good filtration F used to calculate it and is additive over short exact sequences of A-modules. The multiplicity also satisfies the following property: $m_P(M)$ is finite and non-zero if and only if $M \neq 0$ and P is a prime ideal minimal over $I(M)$. For details see [5, Corollary 1.3; 1; 6].

We may now define the main object of study in this paper. Let $k \geq 4$, and $f \in S(k)$ be a generic polynomial. Denote by $\mathscr{M}(f)$ the full subcategory of all finitely generated left A-modules M such that $\mathrm{Ch}(M) = Z(f)$. This category is closed under submodules, quotients, and extensions. If P is the prime ideal of S generated by f, and if M is an object of $\mathscr{M}(f)$, set $m(M) = m_P(M)$. This will be called the multiplicity of M. The length of an object M in $\mathscr{M}(f)$ is bounded above by its multiplicity $m(M)$ which is always finite. These are properties that $\mathscr{M}(f)$ shares with the category of holonomic modules. However, if M is an object in $\mathscr{M}(f)$ then $d(M) = \dim(Z(f)) = 2n - 1$. Hence M has codimension one, and cannot be holonomic.

We end this section with a very useful division lemma. If $d \in A$ is an operator of order k, put $\sigma(d) = \sigma_k(d)$. This is called the *principal symbol* of d.

LEMMA 2.3. *Let $a, d \in A$. Then either $a \in Ad$, or there exists $q \in A$ such that $\sigma(d)$ does not divide $\sigma(a - q \cdot d)$.*

Proof. If $\sigma(d)$ does not divide $\sigma(a)$ there is nothing to do. Suppose that $\sigma(d)$ divides $\sigma(a)$, but that $a \notin Ad$. Then there exists $q_1 \in A$ such that $\sigma(a) = \sigma(q_1) \cdot \sigma(d)$. Hence $\mathrm{ord}(a - q_1 \cdot d) < \mathrm{ord}(a)$. By induction, there exists $q_2 \in A$, such that $\sigma(a - (q_1 + q_2) \cdot d)$ is not divisible by $\sigma(d)$. The result follows if we set $q = q_1 + q_2$.

3. IRREDUCIBLE MODULES

Let $k \geq 4$ be an integer and $P \in S(k)$ be a generic polynomial, which will be kept fixed throughout this section. In Section 2 we saw that if $d \in A$ satisfies $\sigma(d) = P$, then A/Ad is irreducible. We now show that this construction yields infinitely many irreducible objects in $\mathscr{M}(P)$.

THEOREM 3.1. *Let d, d' be two distinct elements of A both with principal symbol P. Then the irreducible modules A/Ad and A/Ad' are not isomorphic.*

Proof. The proof is by contradiction. Assume that there exists an isomorphism $\phi: A/Ad \to A/Ad'$. Since A/Ad is irreducible, it is generated by $1 + Ad$. Thus the isomorphism is completely determined by the image of this element, say $\phi(1 + Ad) = a + Ad'$. By Lemma 2.3, we may assume that $P = \sigma_k(d')$ does not divide $\sigma(a)$. Note also that one must have $a \notin \mathbb{C}$, since $d \neq d'$.

Since $\phi(Ad) \subseteq Ad'$, we must have $d \cdot a = b \cdot d'$, for some $b \in A$. From $\sigma_k(d) = \sigma_k(d')$ it follows that $d' = d + h$, for some $h \in A(k-1)$. We end up with the equation

$$d \cdot a = b \cdot (d + h). \tag{3.2}$$

Notice that a and b must have the same order, say $\operatorname{ord}(b) = \operatorname{ord}(a) = m$. Then taking symbols of order $m + k$ on both sides of Eq. (3.2) we get $\sigma_k(d) \cdot \sigma_m(a) = \sigma_k(d) \cdot \sigma_m(b)$. Hence $\sigma_m(a) = \sigma_m(b)$, and $a = b + c$, for some $c \in A(m-1)$. Substituting this into Eq. (3.2), one has $d \cdot (b + c) = b \cdot (d + h)$, which can be written in the form

$$[d, b] = b \cdot h - d \cdot c. \tag{3.3}$$

The left hand side of (3.3) has order less than or equal to $m + k - 2$, while the right hand side has order at most $m + k - 1$. Suppose that $\operatorname{ord}(c) = m - 1$. Taking symbols of order $m + k - 1$ in (3.3), we get $\sigma_m(b) \cdot \sigma_{k-1}(h) = \sigma_k(d) \cdot \sigma_{m-1}(c)$. But $\sigma_k(d) = P$ is generic, therefore irreducible; hence it must divide either $\sigma_m(b)$ or $\sigma_{k-1}(h)$. Neither of these is possible: $\sigma_m(b) = \sigma_m(a)$ is not divisible by P by hypothesis, and $\sigma_{k-1}(h)$ has smaller degree than P. Therefore we must have $\operatorname{ord}(c) \leq m - 2$. Consequently, $\operatorname{ord}(h) \leq k - 2$. Now, taking symbols of order $m + k - 2$ in (3.3), we have

$$\sigma_{m+k-2}([d, b]) = \sigma_m(b) \cdot \sigma_{k-2}(h) - \sigma_k(d) \cdot \sigma_{m-2}(c).$$

This equation is equivalent to

$$\{P, \sigma_m(b)\} = \sigma_m(b) \cdot \sigma_{k-2}(h) - P \cdot \sigma_{m-2}(c).$$

By Lemma 2.2, the last equation implies that P divides $\sigma_m(b) = \sigma_m(a)$, a contradiction.

Of course all these modules have multiplicity one. We now show that $\mathscr{M}(P)$ contains irreducible modules of multiplicity m for every $m \geq 1$.

PROPOSITION 3.4. *Suppose that $Q \in S(mk-1)$ is not divisible by P. Choose $d, b \in A$ such that $\sigma(d) = P$ and $\sigma(b) = Q$. The left ideal $A(d^m + b)$ is maximal in A.*

Proof. Suppose not, and let J be a proper left ideal of A such that $A(d^m + b) \subset J$. Notice that because $Z(P)$ is a minimal involutive homogeneous variety, we must have that $\sigma(J) \subseteq SP$. Since $m(A/A(d^m + b)) = m$, one has that $s = m(A/J) \leq m - 1$. Hence J contains an element of the form $ad^s + c$, where a has order r, c has order $\leq sk + r - 1$, and P does not divide $\sigma(a)$. The element g defined by

$$g = d^{m-s}(ad^s + c) - a(d^m + b) = [d^{m-s}, a] \cdot d^s + d^{m-s}c - ab$$

is contained in J. But $[d^{m-s}, a] \cdot d^s$ has order $\leq mk + r - 2$, while both $d^{m-s}c$ and ab have order $\leq mk + r - 1$. Hence g has order $\leq mk + r - 1$. Taking symbols of this order, one concludes that $\sigma_{mk+r-1}(d^{m-s}c - ab)$ belongs to $\sigma(J) \subseteq SP$. But

$$\sigma_{mk+r-1}(d^{m-s}c - ab) = \sigma_k(d)^{m-s} \cdot \sigma_{sk+r-1}(c) - \sigma_r(a) \cdot \sigma_{mk-1}(b).$$

Since the left hand side is divisible by P, and $\sigma(d) = P$, it follows that P divides $\sigma_r(a) \cdot \sigma_{mk-1}(b)$. Since P is irreducible, it must divide one or other of these factors, a contradiction.

COROLLARY 3.5. *Suppose that $Q \in S(mk-1)$ is not divisible by P. Choose d, $b \in A$ such that $\sigma(d) = P$ and $\sigma(b) = Q$. The module $M = A/A(d^m + b)$ is irreducible of multiplicity m.*

We now show that the construction of Corollary 3.5 produces an infinite family of irreducible modules of multiplicity m.

PROPOSITION 3.6. *Suppose that $d \in A$ satisfies $\sigma(d) = P^m$. If b is an element of A of order $mk - 1$ whose principal symbol is not divisible by P then A/Ad is not isomorphic to $A/A(d + b)$.*

Proof. The proof is by contradiction. Suppose that ϕ: $A/Ad \to A/A(d + b)$ is an isomorphism. Let $\phi(1 + Ad) = a + A(d + b)$. Clearly $a \notin \mathbb{C}$; and we may also assume that P^m does not divide $\sigma(a)$, by Lemma 2.3. Hence there exists $c \in A$ such that $d \cdot a = c(d + b)$. Suppose that a has order r. Then c too must have order r, and taking symbols one concludes that $\sigma_r(a) = \sigma_r(c)$. Thus we may write $a = c + c'$, where c' has order $< r$. Hence $d(c + c') = da = c(d + b)$; which is equivalent to

$$[d, c] = cb - dc'. \tag{3.7}$$

The order of the left hand side of this equation is at most $mk + r - 2$. On the other hand the term cb in the right hand side has order $mk + r - 1$ and taking symbols of this order in (3.7), one has that

$$0 = \sigma_r(c)\sigma_{mk-1}(b) - \sigma_{mk}(d)\sigma_{r-1}(c').$$

Thus $P^m = \sigma_{mk}(d)$ divides $\sigma_r(c)\sigma_{mk-1}(b)$. Since P is irreducible and does not divide $\sigma_{mk-1}(b)$, it must follow that P^m divides $\sigma_r(c)$, a contradiction.

The contents of Propositions 3.4 and 3.6 are neatly summed up in the statement of the following theorem.

THEOREM 3.8. *Let m be a positive integer. The category $\mathcal{M}(P)$ contains infinitely many non-isomorphic irreducible objects of multiplicity m.*

4. EXTENSIONS

In this section we calculate the first extension groups for some irreducible modules in $\mathcal{M}(P)$. As in the previous section, $k \geq 4$ is an integer and $P \in S(k)$ is a generic polynomial. We begin with a review of some basic facts about Ext-groups.

Let $d \in A$, and J be a left ideal of A. Consider the exact sequence of A-modules

$$0 \to Ad \to A \to A/Ad \to 0.$$

Applying Hom$(\cdot, A/J)$ to it we get the long exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}(A/Ad, A/J) \to \operatorname{Hom}(A, A/J) \to \operatorname{Hom}(Ad, A/J) \\ &\to \operatorname{Ext}^1(A/Ad, A/J) \to 0. \end{aligned}$$

Since $\operatorname{Hom}(A, A/J) \cong \operatorname{Hom}(Ad, A/J) \cong A/J$, the last three terms of this sequence become

$$A/J \xrightarrow{\psi} A/J \to \operatorname{Ext}^1(A/Ad, A/J) \to 0,$$

where ψ is the map $\psi(a + J) = d \cdot a + J$. Hence the isomorphism of vector spaces:

$$\operatorname{Ext}^1(A/Ad, A/J) \cong \operatorname{Coker}(\psi) = A/(J + d \cdot A).$$

THEOREM 4.1. *Let $d, d' \in A$ be such that $\sigma_k(d) = \sigma_k(d') = P$. Then the vector space $\operatorname{Ext}^1(A/Ad, A/Ad')$ has dimension greater than or equal to $\delta(k) = \binom{2n+k-3}{2n}$.*

Proof. According to the calculations that precede this theorem, it is enough to show that the dimension of the vector space $A/(Ad' + dA)$ is greater than or equal to $\delta(k)$. But $\delta(k) = \dim_{\mathbb{C}} A(k-3)$. Thus the theorem will follow if we show that

$$A(k-3) \cap (Ad' + dA) = 0.$$

The proof is by contradiction. Suppose that $b \in A(k-3)$, and that

$$b = a' \cdot d' + d \cdot a \tag{4.2}$$

for some $a, a' \in A$. Without loss of generality we may assume that $\operatorname{ord}(a) = m \geq \operatorname{ord}(a')$ and that $P = \sigma_k(d')$ does not divide $\sigma(a)$. The last assertion follows from Lemma 2.3.

Taking symbols of order $m + k$ in (4.2), we get

$$\sigma_{m+k}(b) = \sigma_m(a') \cdot \sigma_k(d') + \sigma_k(d) \cdot \sigma_m(a).$$

Since $\operatorname{ord}(b) \leq k - 3 < k + m$, we have that $\sigma_{m+k}(b) = 0$. But $\sigma_k(d) = \sigma_k(d')$; thus $\operatorname{ord}(a) = \operatorname{ord}(a')$, and $\sigma_m(a) = -\sigma_m(a')$. Hence $a' = -a + c$, for some $c \in A(m-1)$, and $d' = d + h$ for some $h \in A(k-1)$. Substituting in Eq. (4.2), one has $b = (-a + c) \cdot (d + h) + d \cdot a$. Equivalently,

$$b = [d, a] + (c \cdot d' - a \cdot h). \tag{4.3}$$

Since $\operatorname{ord}([d, a]) \leq m + k - 2$, it follows by taking symbols of order $m + k - 1$ in (4.3), that

$$0 = \sigma_{m-1}(c) \cdot \sigma_k(d') + \sigma_m(a) \cdot \sigma_{k-1}(h).$$

But $\sigma_k(d') = P$ is a generic polynomial, hence irreducible. Thus P must divide either $\sigma_m(a)$ or $\sigma_{k-1}(h)$. However, $\sigma_{k-1}(h)$ has smaller degree than P, and P does not divide $\sigma_m(a)$ by hypothesis. Hence $\operatorname{ord}(c) \leq m - 2$ and $\operatorname{ord}(h) \leq k - 2$.

Finally, taking symbols of order $m + k - 2$ in (4.3), we get

$$0 = \{P, \sigma_m(a)\} + \sigma_{m-2}(c) \cdot P + \sigma_m(a) \cdot \sigma_{k-2}(h)$$

which, by Lemma 2.2, implies that P divides $\sigma_m(a)$, a contradiction.

Theorem 4.1 may be improved when $d = d'$. In this case we show that the dimension of $\operatorname{Ext}^1(A/Ad, A/Ad)$ is actually infinite. The proof is similar, but technically more elaborate. We begin with a lemma.

LEMMA 4.4. *Let $V(t) = \{S(t), P\} + S(t-2) \cdot P$. Then $V(t)$ is a vector subspace of $S(t + k - 2)$ and, for large t,* $\dim V(t) < \dim S(t + k - 2)$.

Proof. Consider the map ϕ: $S(t) \oplus S(t-2) \to S(t+k-2)$, defined by $\phi(f,g) = h_P(f) + gP$. This is clearly a $\mathbb{C}$-vector space homomorphism, and its image is $V(t)$. If $(f,g) \in \mathrm{Ker}(\phi)$, then $\{P, f\} = -gP$. By Lemma 2.2 it follows that f is a multiple of P. Hence $\mathrm{Ker}(\phi) \cong S(t-k)$.

We assume that there exists an infinite set $\mathbb{I} \subseteq \mathbb{Z}$ such that $V(t) = S(t+k-2)$ for all $t \in \mathbb{I}$, and aim at a contradiction. If $t \in \mathbb{I}$, we have an exact sequence

$$0 \to S(t-k) \to S(t) \oplus S(t-2) \to S(t+k-2) \to 0.$$

Let $p(t) = \dim_{\mathbb{C}} S(t)$. This is a polynomial function of t. Since dimension is additive on exact sequences, we have that

$$p(t) + p(t-2) = p(t-k) + p(t+k-2)$$

for all $t \in \mathbb{I}$. Put $q(t) = p(t) - p(t-k)$. The above equation may be rewritten as $q(t) = q(t+k-2)$. Since $q(t)$ is a polynomial, this can only hold for infinitely many values of t if $q(t)$ is identically zero. Hence $p(t) = p(t-k)$ holds for all t. Thus $p(t)$ must also be identically zero, a contradiction. We conclude that there exists some positive integer N such that $V(t)$ is a proper subspace of $S(t+k-2)$ for all $t > N$.

THEOREM 4.5. *If $d \in A(k)$ satisfies $\sigma_k(d) = P$, then the vector space $\mathrm{Ext}^1(A/Ad, A/Ad)$ is infinite dimensional.*

Proof. Let m be an integer. By Lemma 4.4, there exists $N > 0$ so that $S(m) \setminus V(m-k+2) \neq \emptyset$, whenever $m > N$. We have seen that $\mathrm{Ext}^1(A/Ad, A/Ad) \cong A/(Ad + dA)$; thus to prove the theorem it is enough to show that if $b \in A(m)$ and $\sigma_m(b) \in S(m) \setminus V(m-k+2)$, then $b \notin Ad + dA$.

We prove this last statement by contradiction. Without loss of generality, let $m > k$. Suppose that

$$b = a \cdot d + d \cdot a', \tag{4.6}$$

for some $a, a' \in A$. We may assume, by Lemma 2.3, that $P = \sigma_k(d)$ does not divide $\sigma(a)$. Let $\max = \max\{\mathrm{ord}(ad), \mathrm{ord}(da')\}$. Clearly $\max \geq \mathrm{ord}(b)$.

Assume first that $\max = \mathrm{ord}(b)$ and that $\mathrm{ord}(a) \geq \mathrm{ord}(a')$. Then $\mathrm{ord}(a) = m - k$. Applying symbols of order m to (4.6), one has

$$\sigma_m(b) = \sigma_{m-k}(a) \cdot \sigma_k(d) + \sigma_k(d) \cdot \sigma_{m-k}(a').$$

Hence $\sigma_m(b) \in S(m-k) \cdot P \subseteq V(m-k+2)$, which contradicts the choice of b. We arrive at a similar conclusion if $\mathrm{ord}(a') \geq \mathrm{ord}(a)$.

Suppose next that $\max > \mathrm{ord}(b)$. Let $\mathrm{ord}(a) = t$. If $\mathrm{ord}(a) > \mathrm{ord}(a')$, then applying symbols of order $k + t$ in (4.6), one has

$$0 = \sigma_{k+t}(a \cdot d + d \cdot a') = \sigma_{k+t}(a \cdot d) = \sigma_k(d) \cdot \sigma_t(a)$$

which is not possible. Similar results apply if $\operatorname{ord}(a') > \operatorname{ord}(a)$. Therefore, one must have $t = \operatorname{ord}(a) = \operatorname{ord}(a')$. Once again

$$0 = \sigma_{k+t}(a \cdot d + d \cdot a') = \sigma_t(a) \cdot \sigma_k(d) + \sigma_k(d) \cdot \sigma_t(a').$$

Thus $\sigma_t(a) = -\sigma_t(a')$, and we may write $a' = -a + c$, for some $c \in A(t-1)$. Substituting in (4.6), one obtains

$$b = [a, d] + d \cdot c. \tag{4.7}$$

But $\operatorname{ord}([a, d]) \leq k + t - 2$. If $\operatorname{ord}(c) = t - 1$, then

$$\sigma_{k+t-1}(b) = \sigma_{k+t-1}(d \cdot c) = \sigma_k(d) \cdot \sigma_{t-1}(c).$$

If $\sigma_{k+t-1}(b) \neq 0$, then we have a contradiction with the choice of b; on the other hand, if the symbol is zero, then it implies that $\sigma_{t-1}(c) = 0$, contradicting $\operatorname{ord}(c) = t - 1$. Hence $\operatorname{ord}(c) \leq t - 2$, and $\operatorname{ord}(b) \leq k + t - 2$.

Applying symbols of order $k + t - 2$ in (4.7), we get

$$\sigma_{k+t-2}(b) = \sigma_{k+t-2}([a, d]) + \sigma_{t-2}(c) \cdot \sigma_k(d)$$

which is equivalent to

$$\sigma_{k+t-2}(b) = \{\sigma_t(a), P\} + \sigma_{t-2}(c) \cdot P. \tag{4.8}$$

If $\operatorname{ord}(b) < k + t - 2$, then (4.8) implies that $\{\sigma_t(a), P\} = -\sigma_{t-2}(c) \cdot P$. By Lemma 2.2 it follows that P divides $\sigma_t(a)$, which is not possible. Hence we must have that $\operatorname{ord}(b) = k + t - 2$. In this case $t = m - k + 2$, and we get that $\sigma_m(b) = \{\sigma_{m-k+2}(a), P\} + \sigma_{m-k}(c) \cdot P \in V(m - k + 2)$, a contradiction.

5. IDEALS AND REALITIES

In this section we collect a few miscellaneous results related to those proved in the previous section. First of all let us consider the relation between modules in $\mathscr{M}(P)$ and projective ideals of A. This is detailed in the next proposition.

PROPOSITION 5.1. *If I is a projective left ideal of A, then the module A/I has dimension $2n - 1$. On the other hand, if $P \in S(k)$ is a generic polynomial and if M is an object in $\mathscr{M}(P)$, then $M \cong A/J$, where J is a projective left ideal of A.*

Proof. If I is a proper projective left ideal of A then the short exact sequence $0 \to I \to A \to A/I \to 0$ is a projective resolution for A/I. Hence

$j(A/I) = \min\{j\colon \operatorname{Ext}^j(A/I, A) \neq 0\} = 1$. But from [3, V.2.2.2] we have that $j(A/I) + d(A/I) = 2n$. Thus $d(A/I) = 2n - 1$.

Suppose now that M is a module in $\mathscr{M}(P)$. Since M has finite multiplicity, it must have finite length; hence M is cyclic by [2, 1.8.18]. Thus there exists an ideal J of A such that $M \cong A/J$. But from [1, Proposition 5], we have that $\operatorname{Ext}^j(M, A) = 0$ for $j \neq 1$. Thus J must be projective.

We now show how Theorem 3.1 can be used to construct projective non-cyclic ideals of A. This follows an idea of Stafford in [9].

THEOREM 5.2. *Let $k \geq 4$ be an integer and $d \in A(k)$ be such that $\sigma(d) \in S(k)$ is a generic polynomial. If $a \in A \setminus (\mathbb{C} + A \cdot d)$, then the left ideal $I(a, d) = \{x \in A\colon x \cdot a \in A \cdot d\}$ is a non-cyclic projective left ideal of A.*

Proof. By definition, $I = I(a, d)$ is the kernel of the map

$$\begin{aligned} \phi\colon A &\to A/Ad \\ x &\mapsto x \cdot a + Ad. \end{aligned}$$

Thus $A/I \cong A/Ad$. By Proposition 5.1, I is a projective left ideal. We must prove that I is not cyclic.

Suppose, by contradiction, that I is cyclic. Then $I = A \cdot c$, for some $c \in A$. Since $\operatorname{Ch}(A/I) = Z(\sigma(c))$ and $\operatorname{Ch}(A/Ad) = Z(\sigma(d))$ are equal, it follows that $\operatorname{rad}(\sigma(c)) = (\sigma(d))$. Comparing multiplicities, we have that $1 = m(A/Ad) = m(A/I)$. This implies that $c = d + h$, for some $h \in A(k-1)$. By Theorem 3.1, the isomorphism above cannot hold if $h \neq 0$. Hence $c = d$, and ϕ is an endomorphism of A/Ad. By Quillen's Lemma, we must have that $a \in \mathbb{C} + A \cdot d$, a contradiction.

This theorem has a geometrical interpretation, as follows. Let k, m be positive integers. The set $A(m) \times S(k)$ has a natural structure of affine space. According to Theorem 5.2, if (a, P) is generic in $A(m) \times S(k)$ and $\sigma(d) = P$, then the left ideal $I(a, d)$ is projective and non-cyclic. Contrast this with the result of Bernstein and Lunts: if P is generic in $S(k)$ and $\sigma(d) = P$, then any left ideal that contains d is cyclic.

In Section 4 we calculated Ext-groups of modules in $\mathscr{M}(P)$; in the next theorem we consider $\operatorname{Ext}^1(M, N)$ when M and N have different characteristic varieties and show that it must have infinite dimension. The proof depends on a technical lemma.

LEMMA 5.3. *Let $P \in S(k)$ and $Q \in S(m)$. If $m, k > 0$ are integers, then for large enough t, $\dim_{\mathbb{C}}(S(t-k) \cdot P + S(t-m) \cdot Q) < \dim_{\mathbb{C}}(S(t))$.*

The proof follows the general argument of Lemma 4.4 and shall be omitted.

THEOREM 5.4. *Let $m, k \geq 4$ be integers and $d, d' \in A$ be such that $\sigma(d) \in S(k)$ and $\sigma(d') \in S(m)$ are irreducible polynomials. The vector space* $\mathrm{Ext}^1(A/Ad, A/Ad')$ *is infinite dimensional.*

Proof. Recall that $\mathrm{Ext}^1(A/Ad, A/Ad') \cong A/(A \cdot d' + d \cdot A)$. By Lemma 5.3, there exists $N > 0$, such that for all $t > N$, we have $S(t) \setminus (S(t-k) \cdot \sigma_k(d) + S(t-m) \cdot \sigma_m(d')) \neq \emptyset$. Choose $h \in A(t)$, such that $\sigma_t(h) \in S(t) \setminus (S(t-k) \cdot \sigma_k(d) + S(t-m) \cdot \sigma_m(d'))$. It is enough to show that $h \notin Ad' + dA$.

Suppose, by contradiction, that

$$h = a'd' + da \tag{5.5}$$

for some $a, a' \in A$. By Lemma 2.3 we may assume that $\sigma(d)$ does not divide $\sigma(a')$. If $\mathrm{ord}(a'd') < \mathrm{ord}(da) = s$ and $s > t$, then taking symbols of order s in (5.5), one gets $\sigma_k(d) \cdot \sigma_{s-k}(a) = 0$, a contradiction. Similar results apply if $\mathrm{ord}(da) < \mathrm{ord}(a'd')$, and $\mathrm{ord}(a'd') > t$.

If $\mathrm{ord}(da) = \mathrm{ord}(a'd') > t$, then applying symbols of order s again, one has

$$0 = \sigma_k(d) \cdot \sigma_{s-k}(a) + \sigma_m(d') \cdot \sigma_{s-m}(a').$$

Since $\sigma_k(d)$ is irreducible, we conclude that it must divide $\sigma_m(d')$ or $\sigma_{s-m}(a')$, both of which contradict the hypotheses.

We are left with the possibility that $\max\{\mathrm{ord}(da), \mathrm{ord}(a'd')\} = t$. Then taking symbols of order t one obtains

$$\sigma_t(h) = \sigma_k(d)\sigma_{t-k}(a) + \sigma_m(d') \cdot \sigma_{t-m}(a')$$

which, once again, contradicts the choice of h, thus proving the theorem.

We end with a question. As usual, let $k \geq 4$ be an integer, and $P \in S(k)$ be a generic polynomial.

Problem 5.6. Are all irreducible objects of multiplicity 1 in $\mathscr{M}(P)$ of the form A/Ad, for some $d \in A$ with $\sigma(d) = P$?

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