ALGEBRAIC SOLUTIONS OF HOLOMORPHIC FOLIATIONS: AN ALGORITHMIC APPROACH

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Abstract. We present two algorithms that can be used to check whether a given holomorphic foliation of the projective plane has an algebraic solution, and discuss the performance of their implementations in the computer algebra system Singular.

1. Introduction

The study of algebraic solutions of differential equations of the first order and the first degree over the complex projective plane $\mathbb{P}^2$ goes back to the work of G. Darboux in the 1870s. In [11] Darboux showed that if an equation of this kind has enough algebraic solutions then it must have a first integral. In 1891, Poincaré [18] pointed out that in order to find an explicit algebraic solution to such an equation it would be enough to find an upper bound on the degree of the solution in terms of the degree of the polynomials that define the equation. Indeed, if the equation is defined by polynomials of degree less than or equal to 2, then it always has solutions of degree 1, a fact already known to Darboux.

In the twentieth century the results of Darboux and Poincaré were reworked as part of the theory of holomorphic foliations. The search for bounds on the degree of the solution is now known as *Poincaré's Problem*, and many such bounds have been found; see [6], [5] for example. However, these turned out to be of limited use in solving differential equations in view of the following result of J. P. Jouanolou [14, theoreme 1.1, p. 158].

Theorem 1.1. *A generic foliation of $\mathbb{P}^2$ of degree greater than or equal to 2 does not have any algebraic solutions.*

For the definition of the degree of a foliation see section 2. As part of the proof of this theorem, Jouanolou gave an explicit example of a family of foliations with no algebraic solution. However, although Theorem 1.1 tells us that most foliations do not have algebraic solutions, very few concrete examples (say, with rational coefficients) are known. Moreover, most of these examples are variations on Jouanolou's, and make use of the fact that the singular set of the foliation has a rather large symmetry group. However, a greater variety of concrete examples would help in the study of several problems in the theory of holomorphic foliations. Foremost among these is the problem of the existence of nontrivial minimal sets,

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which has already been approached from a computational point of view in [4]. Foliations without algebraic solutions have also been used to construct families of nonholonomic $\mathcal{D}$-modules, see [8] and [9]. Nevertheless, little is known of the properties of these modules, in part because there are so few concrete examples to be used in exploratory concrete calculations.

A more systematic approach to finding examples of holomorphic foliations without algebraic solutions consists in generating a random foliation of a given degree, and using a computer to check that it does not have an algebraic solution of degree less than or equal to the bound provided by a solution of Poincaré's Problem. This was actually successfully implemented in [7]. However, the computations required in this approach are extremely costly, so that it is in practice limited to foliations of degree 2.

One way to improve the algorithmic approach is to settle for a procedure that will either prove that the foliation does not have any algebraic solutions, or return *I don't know*. This is exactly what we do in this paper. In fact, we propose two such algorithms. The reason why these algorithms are expected to be often successful is the well-known fact that a generic polynomial in one variable with rational coefficients is irreducible over $\mathbb{Q}$. As will be shown in a forthcoming paper, a similar strategy can be used to construct families of foliations without algebraic solutions that are far more general than Jouanolou's.

The plan of the paper is as follows. In section 2 we introduce some basic facts concerning foliations of the complex plane in a suitable way for the applications in later sections. The two algorithms are described and proved to be correct in sections 3 and 4. Finally, in section 5 we discuss our implementations of the algorithms in the computer algebra system SINGULAR [20], and analyse their performance.

2. Foliations of the projective plane

In this section we discuss the basic facts about foliations of the complex projective plane $\mathbb{P}^2$ in a way that is suitable for the applications of the forthcoming sections.

Let $n \geq 0$ be an integer, and denote by x, y and z the homogeneous coordinates of the complex projective plane $\mathbb{P}^2$. A *holomorphic foliation* $\mathcal{F}$ of $\mathbb{P}^2$ is defined by a 1-form $\Omega = Adx + Bdy + Cdz$, where A, B and C are homogeneous polynomials of degree $n+1$ that satisfy the identity $xA + yB + zC = 0$. A *singularity* of $\mathcal{F}$ is a common zero of A, B and C. We denote the set of singularities of $\mathcal{F}$ by $\mathrm{Sing}(\mathcal{F})$ or $\mathrm{Sing}(\Omega)$. If $\mathrm{Sing}(\mathcal{F})$ is finite then we say that $\mathcal{F}$ is *saturated*.

Let U_z be the open set of $\mathbb{P}^2$ defined by $z \neq 0$ and let ω be the dehomogeneization of Ω with respect to z. Restricting the foliation of $\mathbb{P}^2$ defined by Ω to U_z, we obtain the foliation of $\mathbb{C}^2$ defined by ω. Conversely, if $\pi_z : U_z \to \mathbb{C}^2$ is the map given by $\pi_z[x : y : z] = (x/z, y/z)$, then $\Omega = z^k \pi_z^*(\omega)$, where k is chosen so as to clear the poles of $\pi_z^*(\omega)$.

From now on we deal only with a foliation of $\mathbb{C}^2$ defined by a 1-from $\omega = adx + bdy$, where $a, b \in \mathbb{C}[x, y]$. Note that if Ω is as above, then

$$a(x, y) = A(x, y, 1) \quad \text{and} \quad b(x, y) = B(x, y, 1).$$

Moreover, we assume that ω is *saturated*, which means $\gcd(a, b) = 1$. A *singularity* of ω is a common zero of a and b. The set of all the singularities of ω is denoted by $\mathrm{Sing}(\omega)$. It follows from Bézout's theorem that this is a finite set, because we are assuming that ω is saturated. Although $\mathrm{Sing}(\omega)$ need not be equal to $\mathrm{Sing}(\Omega)$, the two sets coincide if $\mathrm{Sing}(\Omega)$ does not intersect the line at infinity L_∞. Indeed, in this case, every zero of A and B is also a zero of C because $xA + yB + zC = 0$. From now on, we assume that the coordinates of $\mathbb{P}^2$ have been chosen so that $\mathrm{Sing}(\Omega) \cap L_\infty = \emptyset$.

As a consequence of this choice of coordinates, we have that the polynomial $xA(x, y, 0) + yB(x, y, 0)$ is identically zero, and that $A(x, y, 0)$ and $B(x, y, 0)$ are nonzero homogeneous polynomials. Since $A(x, y, 0)$ and $B(x, y, 0)$ are equal to the leading homogeneous components of a and b, we conclude that

$$a = yh + a_0 \quad \text{and} \quad b = -xh + b_0,$$

where a_0 and b_0 are polynomials of degree less than or equal to n, and h is homogeneous of degree n. In particular,

$$\deg(a) = \deg(b) = n + 1.$$

The number

$$n = \deg(a) - 1 = \deg(b) - 1,$$

is called the *degree* of ω. We also say that n is the degree of the foliation $\mathcal{F}$ defined by ω on $\mathbb{P}^2$.

Let $f \in \mathbb{C}[x, y]$ be a reduced (square free) polynomial, and consider the algebraic curve C defined by the vanishing of f. We say that C is *invariant* under the foliation $\mathcal{F}$, if C is tangent to the vector field dual to ω at every point outside $\mathrm{Sing}(C) \cup \mathrm{Sing}(\omega)$. This is equivalent to the existence of a polynomial 2-form η such that

$$\omega \wedge df = f\eta.$$

The curve C is also called an *algebraic solution* of $\mathcal{F}$ (or ω). By abuse of notation we also talk of f being invariant under ω. The next proposition characterizes the kind of invariant curve that we can expect a 1-form ω to have if its coefficients are rational numbers. The proof given here is based on [17, proposition 3.3, p. 36].

Proposition 2.1. *If ω has an algebraic solution, then there is a reduced polynomial with rational coefficients which is invariant under ω.*

Proof. Suppose that ω has an algebraic solution of degree $k \geq 1$. Let

$$f = \sum_{i+j \leq k} c^1_{ij} x^i y^j \quad \text{and} \quad g = \sum_{s+t \leq n-1} c^2_{st} x^s y^t$$

be polynomials in x and y, with undetermined coefficients. Let

$$C = \{c^1_{ij}, c^2_{st} : 0 \leq i + j \leq k \ \text{ and } \ 0 \leq s + t \leq n - 1\}$$

and denote by N the number of elements of C. Consider the ideal J generated by the coefficients of the monomials in x and y on the left hand side of

$$a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} - gf = 0. \tag{2.1}$$

J is an ideal of the polynomial ring $\mathbb{Q}[C]$.

Since ω has a solution of degree k, then it has a solution for which $c^1_{i_0 j_0} \neq 0$ for some choice of integers $i_0, j_0 \geq 0$ with $i_0 + j_0 = k$. However, the polynomials of J are homogeneous on the c^1s. So we can assume, without loss of generality, that $c^1_{i_0 j_0} = 1$; which implies that the constant polynomial is not a solution of $J_0 = J|_{c^1_{i_0 j_0} = 1}$.

Now consider the variety $\mathcal{X}$ in $\mathbb{C}^{N-1}$ defined by J_0. Suppose, first, that $\dim(\mathcal{X}) = d > 0$. Then by [19, Theorem 10, p. 52], there exists a finite surjective map

$$\pi : \mathcal{X} \to \mathbb{C}^d.$$

Let $q \in \mathbb{Q}^d$, and consider the fibre $\pi^{-1}(q)$. Since π is onto, it follows that $0 < \sharp\pi^{-1}(q) < \infty$. Moreover, since the polynomials that define $\pi^{-1}(q)$ have rational coefficients, then $\pi^{-1}(q) \subset \overline{\mathbb{Q}}^{N-1}$. In particular, the coordinates of the points of $\pi^{-1}(q)$ are algebraic numbers. Therefore, these coordinates must all be contained in a finite normal extension K of $\mathbb{Q}$. Thus, by the definition of $\mathcal{X}$, a point of $\pi^{-1}(q)$ corresponds to a pair of polynomials $f, g \in K[x, y]$ that satisfy (2.1).

Suppose now that $\dim(\mathcal{X}) = 0$. In this case, applying the same argument to $\mathcal{X}$ itself, instead of $\pi^{-1}(q)$, we conclude that there exist polynomials $f, g \in K[x, y]$ that satisfy (2.1), where K is a normal extension of $\mathbb{Q}$.

In either case, let G be the Galois group of K over $\mathbb{Q}$. Since a and b have rational coefficients, it follows that $\sigma(f)$ and $\sigma(g)$ also satisfy (2.1) for all $\sigma \in G$. Therefore, $F = \prod_{\sigma\in G} \sigma(f)$ is also a solution of (2.1). However, F is invariant under G, hence its coefficients must be rational. Thus, the squarefree part of F is a reduced polynomial with rational coefficients that is an algebraic solution of ω, which proves the proposition. □

We now turn to the definition of the characteristic exponents, which will play a very important rôle in both of our algorithms. But, first, we fix the hypotheses that will be in force for the remainder of the section:

Hypotheses 2.2. *Take $\mathcal{F}$ to be a foliation of $\mathbb{P}^2$ determined by a 1-form $\omega = adx + bdy$, where $a, b \in \mathbb{Q}[x, y]$, and assume that* $\mathrm{Sing}(\mathcal{F}) \cap L_\infty = \emptyset$.

Let $p \in \mathrm{Sing}(\omega)$. The 1-jet at p of the vector field dual to ω is

$$J_\omega(p) = \begin{bmatrix} \partial b/\partial x & \partial b/\partial y \\ -\partial a/\partial x & -\partial a/\partial y \end{bmatrix}$$

We say that $\mathcal{F}$ is *nondegenerate* at p if $\det(J_\omega(p)) \neq 0$. In this case, the eigenvalues λ_1 and λ_2 of $J_\omega(p)$ are both nonzero, and the quotient λ_1/λ_2 and its reciprocal are the *characteristic exponents* of ω at p. Let

$$\rho_\omega(p) = \frac{\mathrm{trace}(J_\omega(p))^2}{\det(J_\omega(p))}.$$

An easy computation shows that $\rho_\omega(p)$ is related to the characteristic exponents by the formula

$$\rho_\omega(p) = \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} + 2. \tag{2.2}$$

The set of all complex numbers that are characteristic exponents of $\mathcal{F}$ at one of its singularities will be denoted by $\mathrm{Exp}(\mathcal{F})$ or $\mathrm{Exp}(\omega)$. For a proof of the next proposition see [14, Proposition 4.1, p. 126], [21, Lemma 5.1, p. 156] and [6, Theorem 1, p. 891].

Proposition 2.3. *If C is a reduced algebraic curve that is invariant under ω, then* $\mathrm{Sing}(\omega) \cap C \neq \emptyset$. *Moreover, if* $\mathrm{Exp}(\omega) \cap \mathbb{Q} = \emptyset$ *then all the singularities of the projectivization $\overline{C}$ of C are nodes and*

$$\deg(\overline{C}) \leq \deg(\mathcal{F}) + 2.$$

Given a singular point $p \in \mathbb{C}^2$ of ω, let $V_{p,\lambda}$ be the eigenspace of $J_\omega(p)$ with respect to the eigenvalue λ. If C is a reduced algebraic curve invariant under ω, and ω is nondegenerate at all $p \in \mathrm{Sing}(\omega)$, set

$$\mathrm{Exp}(\omega, C) = \{\lambda_1/\lambda_2 \in \mathrm{Exp}(\omega) : V_{p,\lambda_2} \cap T_pC \neq 0 \text{ for some } p \in \mathrm{Sing}(\omega) \cap C\}.$$

The next theorem is an immediate consequence of the Camacho-Sad Index Theorem; see [3, Theorem 2, p. 37].

Theorem 2.4. *Let C be a reduced algebraic curve of degree d invariant under ω. If all the singularities of C are nodes, and ω is nondegenerate at all $p \in \mathrm{Sing}(\omega)$, then*

$$\sum_{q \in \mathrm{Exp}(\omega, C)} q = d^2 - 2\delta$$

where δ is the number of nodes of C.

The final result of this section is a corollary of a famous theorem of Baum and Bott [2, Theorem 1, p. 280], although the first half of the result was originally proved by Darboux [11, p.84]. For a direct proof in this special case see [22, Theorem 1.1, p. 150] or [3, Theorem 1, p. 34]. Before we state the theorem, we must introduce some notation. If p is a singularity of ω, define the *multiplicity* $\mu_p(\omega)$ of ω at a singularity p to be the intersection number of a and b at p. In particular, $\mu_p(\omega) = 1$ if and only if ω is nondegenerate at p.

Theorem 2.5. *Let ω be a 1-form of degree n that satisfies $\mathrm{Sing}(\omega) \cap L_\infty = \emptyset$, then*

$$\sum_{p \in \mathrm{Sing}(\omega)} \mu_p(\omega) = n^2 + n + 1. \tag{2.3}$$

Moreover, if ω is nondegenerate at all of its singular points, then

$$\sum_{p \in \mathrm{Sing}(\omega)} \rho_\omega(p) = (n+2)^2. \tag{2.4}$$

The following result is an immediate consequence of the theorem, and will be useful in the coming sections.

Corollary 2.6. *Let ω be a 1-form of degree n that satisfies $\mathrm{Sing}(\omega) \cap L_\infty = \emptyset$, then ω has $n^2 + n + 1$ singularities, counted with multiplicity, all of which belong to the open set $z \neq 0$. Conversely, if ω has $n^2 + n + 1$ distinct singularities at $z \neq 0$ then $\mathrm{Sing}(\omega) \cap L_\infty = \emptyset$.*

3. The first algorithm

Let a and b be polynomials of degree $n+1$ in $\mathbb{Q}[x, y]$, and consider the 1-form $\omega = adx + bdy$. Let $g_0(x)$ be a generator of the ideal $(a, b) \cap \mathbb{Q}[x]$. Suppose that g_0 is irreducible over $\mathbb{Q}$ of degree n^2+n+1. Note that these conditions imply that the foliation induced by ω has $n^2 + n + 1$ distinct singular points, all of which belong to the open set $z \neq 0$. Moreover, L_∞ cannot be invariant under ω by Proposition 2.3. Therefore, $xa_{n+1} + yb_{n+1} = 0$, as we have seen in section 2. The proof of the next theorem is inspired on that of [6, Theorem, p. 90].

Theorem 3.1. *Let $\mathcal{F}$ be a foliation of $\mathbb{P}^2$ determined by a 1-form $\omega = adx + bdy$, where $a, b \in \mathbb{Q}[x, y]$. Assume that:*

1. *$\mathrm{Sing}(\mathcal{F}) \cap L_\infty = \emptyset$;*

(2) *$\mathcal{F}$ has degree $n \geq 2$;*
(3) *g_0 is the generator of the ideal $(a,b) \cap \mathbb{Q}[x]$.*

If g_0 is irreducible over $\mathbb{Q}$ of degree n^2+n+1, then $\mathcal{F}$ does not have any algebraic solutions in $\mathbb{P}^2$.

Proof. Since g_0 is irreducible, it follows that

$$\mathbb{Q}[x]/(g_0) \hookrightarrow \mathbb{Q}[x,y]/\sqrt{(a,b)}.$$

But,

$$n^2+n+1 = \dim_{\mathbb{Q}}(\mathbb{Q}[x]/(g_0)) \leq \dim_{\mathbb{Q}}(\mathbb{Q}[x,y]/\sqrt{(a,b)}) \leq n^2+n+1,$$

so that both algebras have dimension n^2+n+1 over $\mathbb{Q}$. Therefore,

$$\sqrt{(a,b)} = (g_0, y-g_1),$$

where g_1 is a polynomial in $\mathbb{Q}[x]$ of degree at most $\deg(g_0)-1$. The set $\{g_0, y-g_1\}$ is a reduced Gröbner basis of $\sqrt{(a,b)}$ for the lexicographical order with $y > x$. In particular, the singularities of ω are of the form $(x_0, g_1(x_0))$, for some complex root x_0 of g_0.

Let G be the Galois group of g_0 over $\mathbb{Q}$. Since g_0 is irreducible over $\mathbb{Q}$, it follows that G acts transitively on the set of roots of g_0. Hence, it must also act transitively on the set $\mathrm{Sing}(\omega)$, by

$$\sigma(x_0, g_1(x_0)) = (\sigma(x_0), g_1(\sigma(x_0))),$$

for $\sigma \in G$.

Assume now that ω has an algebraic solution. Then, by Proposition 2.1 there exists a reduced polynomial $f \in \mathbb{Q}[x,y]$ that is invariant under ω. Since f and ω are both stable under G, it follows from Proposition 2.3 that

$$\mathrm{Sing}(\omega) \subset \mathcal{Z}(f) = C.$$

We must analise two cases.

FIRST CASE: C is nonsingular at every point of $\mathrm{Sing}(\omega)$.

We have, by hypothesis, that C is nonsingular at every singular point of ω. But, being invariant under ω, the curve C cannot be singular anywhere else. Since $\mathcal{F}$ does not have singularities at L_∞, it follows that the projectivization $\overline{C}$ of C is a nonsingular curve of $\mathbb{P}^2$. Hence, by [14, Proposition 4.1, p. 126] there exists a homogeneous polynomial h, and a homogeneous 1-form η such that

$$\Omega = hdF + F\eta, \tag{3.1}$$

where F and Ω denote the homogeneizations of f and ω with respect to z. Taking into account that the coefficients of Ω have degree $n+1$, we see that $\deg(h) + \deg(F) = n+2$.

However,

$$\mathrm{Sing}(\Omega) = \mathrm{Sing}(\omega) \subseteq Z(F) = \overline{C},$$

which is the projectivization of C. Therefore, by (3.1), hdF vanishes at every singularity p of ω. But, $\overline{C}$ is a nonsingular curve, so that $dF(p) \neq 0$ at every $p \in \overline{C}$. We conclude that $h(p) = 0$ for every $p \in \mathrm{Sing}(\Omega)$. In particular,

$$\#(\overline{C} \cap \mathcal{Z}(h)) \geq n^2+n+1.$$

However, by Bézout's Theorem

$$\#(\overline{C} \cap \mathcal{Z}(h)) = \deg(F)\deg(h) = \deg(F)(n+2-\deg(F)).$$

Moreover, $\deg(F) \leq n+1$ by [14, Proposition 4.1, p. 126], so that

$$\deg(F)(n+2-\deg(F)) < n^2+n+1,$$

whenever $\deg(F) \geq 2$. Thus, $\deg(F) = 1$. But all the singularities of ω are also zeroes of a^h, the homogeneization of the polynomial a with respect to z. Since a^h has degree $n+1$, it follows by Bézout's Theorem that

$$n^2+n+1 \leq \deg(a^h)\deg(F) = \deg(a^h) = n+1,$$

a contradiction. Therefore, ω cannot have a nonsingular invariant curve.

Second case: C is singular at some point $p_0 \in \operatorname{Sing}(\omega)$.

Since f is singular at $p_0 \in \operatorname{Sing}(\omega)$, it follows that $(\nabla f)(p_0) = 0$. But G acts transitively on $\operatorname{Sing}(\omega)$, and f has rational coefficients, so that

$$0 = \sigma((\nabla f)(p_0)) = (\nabla f)(\sigma(p_0)).$$

Therefore, C is singular at every singularity of ω.

We now turn to some properties of ω. We already know that ω has n^2+n+1 distinct singularities. Thus, by Theorem 2.5,

$$\mu_p(\omega) = 1, \text{ for every } \ p \in \operatorname{Sing}(\omega).$$

In particular, ω is nondegenerate at every one of its singularities.

Next, we want to show that ω does not have any rational characteristic exponents. In order to do this, consider the set

$$R = \{\rho_\omega(p) : p \in \operatorname{Sing}(\omega)\}.$$

If ω has a rational exponent, then $R \cap \mathbb{Q} \neq \emptyset$. However, G acts transitively on $\operatorname{Sing}(\omega)$ and since

$$\sigma(\rho_\omega(p)) = \rho_\omega(\sigma(p)),$$

it follows that σ acts transitively on R. Thus, $R \cap \mathbb{Q} \neq \emptyset$, implies that all the elements of R are rational numbers. But rational numbers are stable under G, so that $R = \{q\} \subset \mathbb{Q}$. Hence, by Theorem 2.5, we conclude that

$$(n^2+n+1)q = (n+2)^2.$$

In particular, if e and $1/e$ are the corresponding characteristic exponents, we find that

$$e = \frac{-n^2+2n+2}{2(n^2+n+1)} + \frac{n(n+2)}{2(n^2+n+1)} \cdot \imath\sqrt{3}.$$

But this is not a rational number. Therefore, $\operatorname{Exp}(\omega) \cap \mathbb{Q} = \emptyset$.

Thus, by Proposition 2.3, all the singularities of $\overline{C}$ must be nodes. Since $\overline{C}$ is reduced, it follows from [13, Problem 5-25, p.118] and the inequality of Proposition 2.3 that

$$n^2+n+1 = \sum_{p \in \operatorname{Sing}(\omega)} \frac{m_p(m_p-1)}{2} \leq \frac{n^2+3n+2}{2},$$

where m_p is the multiplicity of $\overline{C}$ at p. But this inequality implies that $n \leq 1$, which is a contradiction. □

We will isolate a consequence of the last part of the proof of Theorem 3.1 for future reference.

Corollary 3.2. *Let ω be a saturated 1-form and let C be a reduced algebraic curve of $\mathbb{C}^2$. If*

- $\mathrm{Sing}(\omega) \subseteq \mathrm{Sing}(C)$, *and*
- $\mathrm{Exp}(\omega) \cap \mathbb{Q} = \emptyset$,

then C cannot be invariant under ω.

These results provides a strategy to check that a given saturated 1-form of degree $n \geq 1$, say $\omega = adx + bdy$, does not have any algebraic invariant curves. All we have to do is check that L_∞ is not invariant under ω, and that the generator g_0 of $(a, b) \cap \mathbb{Q}[x, y]$ is irreducible of degree $n^2 + n + 1$. The desired conclusion follows from Theorem 3.1.

The most obvious way to implement this strategy is to compute a Gröbner basis for (a, b) with respect to the lexicographical order with $x < y$. The polynomial g_0 is one of the elements of this basis. The irreducibility of g_0 can be checked using a factorization algorithm. Moreover, since ω is saturated, the ideal generated by a and b is zero dimensional. Thus we can improve the performance of the procedure using the FGLM algorithm to compute the Gröbner basis. [12], [1, exercise 2.2.8, p. 68]. However, in practice, there is an altogether better approach which consists in using resultants, instead of Gröbner basis, as shown in the following algorithm.

Algorithm 3.3. *Given a 1-form $\omega = adx+bdy$, where $a, b \in \mathbb{Q}[x, y]$ are polynomials of degree $n + 1 \geq 3$, the algorithm returns one of four messages:* `the foliation is not saturated`, `the line at infinity is an algebraic solution`, `there are no algebraic solutions`, *or* `do not know`.

Step 1: *If* $\gcd(a, b) \neq 1$, *stop and return* `the foliation is not saturated`.
Step 2: *If the polynomial $xa_{n+1} + yb_{n+1}$ is nonzero, stop and return* `the line at infinity is an algebraic solution`.
Step 3: *Compute the resultant $R(x)$ of a and b with respect to y.*
Step 4: *If R is reducible or* $\deg(R) < n^2 + n + 1$, *stop and return* `do not know`.
Step 5: *Stop and return* `there are no algebraic solutions`.

Proof. Steps 1 and 2 check that ω is saturated, and that L_∞ is not invariant under ω. In particular, this implies that ω induces a foliation of degree n in $\mathbb{P}^2$. Since

$$R \in (a, b) \cap \mathbb{Q}[x] = (g_0),$$

we conclude from step 4 that $R = g_0$. The result now follows from Theorem 3.1. □

It may be worth pointing out that this algorithm is not in any way weaker than the one originally proposed. After all, if the generator g_0 of $(a, b) \cap \mathbb{Q}[x]$ is irreducible of degree $n^2 + n + 1$, then it must be equal to the resultant R. This follows from the fact that every x-coordinate of a singular point of ω must be a root of R, which is a polynomial of degree less than or equal to $n^2 + n + 1$.

4. The second algorithm

The algorithm discussed in the previous section will work only if the x coordinates of the singular points of the 1-form ω are roots of a polynomial equation that is irreducible over $\mathbb{Q}$. Although this condition is expected to hold generically, it fails often when the polynomials that define ω are sparse, as shown in section 5. However, there is another algorithm, based on Theorem 2.4 that might work even in this case.

Let ω be a 1-form with rational coefficients. Assume that the hypotheses of 2.2 are in force, and that ω is nondegenerate at every one of its singularities. Consider the ideal

$$L = (a, b, t\det(J_\omega) - \operatorname{trace}(J_\omega)^2)$$

of $\mathbb{Q}[x, y, t]$, and let q be the generator of $L \cap \mathbb{Q}[t]$. By equation (2.2), the characteristic exponents of ω must all be roots of the polynomial

$$\hat{q} = u^{(n^2+n+1)} q\left(u + \frac{1}{u} + 2\right).$$

The algorithm depends on the following result.

Proposition 4.1. *Let $f \in \mathbb{Q}[x, y]$ be a reduced polynomial and denote by C the curve defined by $f = 0$. Assume that:*

1. *ω is nondegenerate;*
2. *q is reduced of degree $n^2 + n + 1$,*
3. *q does not have any rational roots, and*
4. *C is invariant under ω.*

Then, there exists a subset S of the set of irreducible factors of $\hat{q}$ over $\mathbb{Q}$ such that $\operatorname{Exp}(\omega, C)$ is the set of all roots of $\Phi = \prod_{\phi \in S} \phi$.

Proof. First of all, note that if q is reduced of degree $n^2 + n + 1$ with no rational roots, then $\hat{q}$ is reduced of degree $2(n^2 + n + 1)$, and also has no rational roots. These are the hypotheses on $\hat{q}$ that are used in the proof of the proposition.

Let $j = 1, 2$. Denote by M_j the ideal of 2×2 minors of the matrix

$$\begin{bmatrix} J_\omega - v_j I \\ \nabla f \end{bmatrix},$$

and by Δ_j the determinant of $J_\omega - v_j I$. Consider the ideals

$$I_1 = (a, b, \Delta_1, \Delta_2, v_1 - uv_2, (v_1 - v_2)w - 1) \text{ and } I_2 = I_1 + M_2$$

of $\mathbb{Q}[x, y, v_1, v_2, u, w]$, and let γ_1 be the generator of $\sqrt{I_1} \cap \mathbb{Q}[u]$.

If $\lambda_1 \neq \lambda_2$ are two eigenvalues of the 1-jet of ω at the singularity (x_0, y_0), then $u_0 = \lambda_1/\lambda_2 \neq 1$ is a characteristic exponent of ω and

$$(x_0, y_0, \lambda_1, \lambda_2, u_0, 1/(\lambda_1 - \lambda_2)) \in \mathcal{Z}(I_1),$$

the set of zeroes of I_1 in $\mathbb{C}^6$. By [10, Lemma 1, Chapter 3, section 2, p. 121], every element of $\operatorname{Exp}(\omega)$ is a root of γ_1. Thus, $\hat{q}$ divides γ_1. Hence,

$$\deg(\hat{q}) \leq \deg(\gamma_1) \leq \dim(\mathbb{Q}[x, y, v_1, v_2, u, w]/\sqrt{I_1}) \leq 2(n^2 + n + 1).$$

It follows, from the hypothesis on $\hat{q}$, that $\gamma_1 = \hat{q}$, and that

$$\deg(\gamma_1) = \dim(\mathbb{Q}[x, y, v_1, v_2, u, w]/\sqrt{I_1}) = 2(n^2 + n + 1).$$

In particular, by [15, Theorem 3.7.23, p. 255] $\sqrt{I_1}$ is in general position with respect to u (or normal u-position in the terminology of [15]).

We must now identify the zeroes of I_2. Suppose that

$$(x_0, y_0, \lambda_1, \lambda_2, u_0, w_0) \in \mathcal{Z}(I_2).$$

Then,

- $(x_0, y_0) \in \mathrm{Sing}(\omega)$,
- $u_0 = \lambda_1/\lambda_2$,
- $\lambda_1 \neq \lambda_2$,
- λ_1 is an eigenvalue of ω at (x_0, y_0), and
- $w_0 = 1/(\lambda_1 - \lambda_2)$.

We must still investigate the condition imposed by the ideal of minors M_2. Two cases can occur. If $\nabla f(x_0, y_0) \neq 0$, then the vanishing of M_2 implies that there exists an eigenvector $v \neq 0$ of λ_2, that is tangent to C. On the other hand, if $\nabla f(x_0, y_0) = 0$ then f is singular at (x_0, y_0), so that $T_pC = \mathbb{C}^2$. In either case, $u_0 \in \mathrm{Exp}(\omega, C)$. Since the converse is clearly true, we conclude that

u_0 is a u-coordinate of a point in $\mathcal{Z}(I_2)$ if and only if $u_0 \in \mathrm{Exp}(\omega, C)$.

Since $I_1 \subseteq I_2$, and $\sqrt{I_1}$ is in general position with respect to u, it follows that so is $\sqrt{I_2}$. Therefore, by [15, Theorem 3.7.25], a complex number is a u-coordinate of a point of $\mathcal{Z}(I_2)$ if and only if it is a root of the generator γ_2 of $\sqrt{I_2} \cap \mathbb{Q}[u]$. But, $I_1 \subseteq I_2$ implies that γ_2 divides $\gamma_1 = \hat{q}$. Since $\hat{q}$ is reduced, the theorem is proved. □

The strategy now consists in showing that Theorem 2.4 cannot be satisfied by any curve with rational coefficients. However, before we do this we must determine the number of nodes of a curve invariant under ω. Suppose that C, $\hat{q}$ and Φ are as in Proposition 4.1. If C is singular then, by Proposition 2.3, all of its singularities are nodes. Let δ be the *number of nodes* of C. Given a polynomial $\phi(u)$, in one variable, let

$$\tilde{\phi} = u^{\deg(\phi)}\phi(1/u).$$

Corollary 4.2. *We have that* $\deg(\gcd(\Phi, \tilde{\Phi})) = 2\delta$.

Proof. Let α be a characteristic exponent of ω at $p \in \mathrm{Sing}(\omega)$. Then, p is a node of C if and only if both α and $1/\alpha$ belong to $\mathrm{Exp}(\omega, C)$. But this is equivalent to α and $1/\alpha$ being roots of Φ. Therefore, the number of roots of Φ whose reciprocal is also a root of Φ is exactly 2δ.

On the other hand, α is a root of Φ if and only if $1/\alpha$ is a root of $\tilde{\Phi}$. Therefore, α is a root of $d = \gcd(\Phi, \tilde{\Phi})$ if and only if both α and $1/\alpha$ are roots of Φ. Of course, in this case $1/\alpha$ is also a root of d. From this remark, and the previous paragraph, we conclude that $\deg(d) = 2\delta$. □

We are now ready to give a step by step description of the second algorithm.

Algorithm 4.3. *Given a* 1*-form* $\omega = adx + bdy$*, where* $a, b \in \mathbb{Q}[x, y]$ *are polynomials of degree* $n + 1 \geq 3$*, the algorithm returns one of four messages:* `the foliation is not saturated`*,* `the line at infinity is an algebraic solution`*,* `there are no algebraic solutions`*, or* `do not know`*.*

Step 1: *If* $\gcd(a, b) \neq 1$*, stop and return* `the foliation is not saturated`*.*

Step 2: *If the polynomial* $xa_{n+1} + yb_{n+1}$ *is nonzero, stop and return* `the line at infinity is an algebraic solution.`

Step 3: *Compute the generator* q *of* $L \cap \mathbb{Q}[t]$*, where*

$$L = (a, b, t\det(J_\omega) - \mathrm{trace}(J_\omega)^2)$$

Step 4: *If* $q = 0$ *or* $\deg(q) < n^2 + n + 1$ *stop and return* `do not know.`

Step 5: *If* $\gcd(q, dq/dt) \neq 1$ *stop and return* `do not know.`

Step 6: *Let*

$$\hat{q} = u^{(n^2+n+1)} q\left(u + \frac{1}{u} + 2\right).$$

Step 7: *Compute the set* T *of factors of* $\hat{q}$ *over* $\mathbb{Q}$.

Step 8: *If* T *contains a polynomial of degree* 1*, stop and return* `do not know.`

Step 9: *For every proper subset* $S \subsetneq T$ *do:*

Find the product Φ_S *of all polynomials in* S.

Let

$$\widetilde{\Phi}_S = u^{\deg(\Phi_S)}\Phi_S(1/u).$$

Compute the coefficients c_m *and* c_{m-1} *of* Φ_S*, where* $m = \deg(\Phi_S)$*, and let*

$$\beta(S) = c_{m-1}/c_m + \deg(\gcd(\Phi_S, \widetilde{\Phi}_S))$$

If $\beta(S)$ *is an integer and a perfect square, stop and return* `do not know.`

Step 10: *Return* `there are no algebraic solutions.`

Proof. As in Algorithm 3.3, steps 1 and 2 merely check that the foliation is saturated, and that the line at infinity is not invariant under ω. In order to apply Proposition 4.1, we must first compute the polynomial $\hat{q}$ (steps 3 and 6), and check that it satisfies the assumptions of the proposition (steps 4, 5 and 8). Since we would have to factor $\hat{q}$ anyway, we prefered to check if it, rather than q, had any rational roots. Note that if ω has a degenerate singularity then $L \cap \mathbb{Q}[t] = \{0\}$, so that $q = 0$. If this is the case the program will stop in Step 4.

Let us now turn to step 9. Suppose that ω has an invariant algebraic curve. Thus, by Proposition 2.1, it must have an invariant algebraic curve C with rational coefficients. But Proposition 4.1 then implies that $\mathrm{Exp}(\omega, C)$ is equal to the set of roots of

$$\Phi_S = \prod_{\phi \in S} \phi = \sum_{j=0}^{m} c_j u^j,$$

where S is a subset of the set T of all factors of $\hat{q}$. Note that we may assume that S is a proper subset of T, for otherwise C would be singular at every point of $\mathrm{Sing}(\omega)$, which has been ruled out by Corollary 3.2. However, since the characteristic exponents of ω are not rational numbers, it follows from Theorem 2.4 and Corollary 4.2 that the sum of the roots of Φ_S is equal to $\deg(C)^2 - \deg(\gcd(\Phi_S, \widetilde{\Phi}_S))$. By Newton's formula this sum is also equal to c_{m-1}/c_m. Hence,

$$\beta(S) = c_{m-1}/c_m + \deg(\gcd(\Phi_S, \widetilde{\Phi}_S))$$

must be an integer and a perfect square. Step 9 checks if this assumption is realised for some proper subset S of T. If it is not, then ω cannot have any invariant algebraic curves, and the proof of the algorithm is complete. $\square$

Even when this algorithm fails, it provides information on the possible solutions of ω. Let C be a curve with rational coefficients that is invariant under ω. Then, $\deg(C)$ must be an integer in the set $\{\sqrt{\beta(S)} : S \subsetneq T\}$. Moreover, the roots of the generator of $(L, \Phi_S) \cap \mathbb{Q}[x]$ are the x coordinates of the points of $\mathrm{Exp}(\omega, C)$. Once one has this information, it is possible to use the method of undetermined coefficients to find the actual solution.

5. Experimental tests

The algorithms described in sections 3 and 4 were implemented using the computer algebra system Singular (version 2.0.5) [20]. From now on we assume that all the 1-forms ω that we will be talking about can be written as

$$\omega = (hy + f)dx + (-xh + g)dy \tag{5.1}$$

where $h \in \mathbb{Q}[x, y]$ is a homogeneous polynomial of degree n and $f, g \in \mathbb{Q}[x, y]$ are polynomials of degree at most n.

In this case, the first algorithm checks only if the system of polynomial equations that defines the singularities of ω is in general position with respect to x. But this property holds generically in the set of 1-forms that we are considering. Therefore, it is not surprising that randomly generated pairs of dense polynomials of type $(hy+f, -xh+g)$ almost always give rise to a foliation that does not have algebraic solutions. Table 1 gives the average time taken by the algorithm to prove that a given pair is in general position in terms of the degree of the corresponding foliation.

All the tests discussed in this section were performed under Windows 2000 running on a micro-computer with an Intel Pentium 4 HT processor of 2.8 GHz, with 512 MB of primary memory. Table 1 summarizes the output of a program that generates 50 pairs of dense polynomials for each degree and computes the total CPU time taken to check that they are all in general position.

Degree of the foliation	Average execution time
2	2ms
5	31ms
10	750ms
15	9,7s
20	1min e 12s
25	6min e 5s
30	19min e 24s

Table 1. Dense polynomials

If we generate sparse, instead of dense, polynomials the algorithm fails rather often. There are two possible reasons for this, either (a) there are singularities at the line at infinity L_∞, or (b) some of the singularities have multiplicity greater than one. We performed an experimental test which consists in generating 50 forms of degree n, for $2 \le n \le 7$. Each time the algorithm failed, we checked whether that happened because of (a) or (b), above. The results are summarized in table 2.

As it stands, Algorithm 3.3 cannot cope with (b); however, (a) at first seems to be only a case of bad luck. Indeed, by changing the coordinates we can easily arrange

Degree of foliation	Number of failures	Singular at L_∞ (a)	Not in general position (b)
2	6	4	2
3	15	12	3
4	13	9	4
5	14	12	2
6	16	13	3
7	13	8	5

TABLE 2. Sparse polynomials

for the line at infinity to be free of singularities of the 1-form. Unfortunately, this simple device does not work, as we proceed to show.

Let ω be a 1-form with rational coefficients, and let Ω be its homogeneization with respect to z. Consider the projective transformation defined by $T[x:y:z] = [x:y:z-\lambda(x,y)]$, where $\lambda \in \mathbb{Q}[x,y]$ is a homogeneous polynomial of degree one. Denote by $\widehat{\omega} = \widehat{a}dx + \widehat{b}dy$ the dehomogeneization of $T^*(\Omega)$ with respect to z. Then,

$$(g_0) = (\widehat{a}, \widehat{b}) \cap \mathbb{Q}[x] \subsetneq (\widehat{a}, \widehat{b}, \lambda) \cap \mathbb{Q}[x] = (f).$$

If

$$\mathrm{Sing}(\Omega) \cap L_\infty \neq \emptyset \text{ and } \mathrm{Sing}(\Omega) \not\subset L_\infty,$$

and all the singularities of $T^*(\Omega)$ are outside L_∞ then $f \neq 1$. Moreover, f is a proper factor of the polynomial g_0 of Theorem 3.1. In particular, g_0 cannot be irreducible. However, g_0 is always a factor of the polynomial R computed by Algorithm 3.3, so the algorithm will fail in this case.

As one might expect, the number of foliations for which the algorithm fails is proportional to the number of vanishing coefficients in a and b, as the next table shows. The data were obtained with a procedure that tests 50 randomly generated foliations of degree 3 of each type. In performing this test we used the SINGULAR function `sparsepoly` which randomly chooses both, the coefficients that are going to be zero, and the size of the nonzero coefficients. This function also allows the user to choose the percentage of vanishing coefficients in the polynomial that it will generate. The dense polynomials were obtained with the help of the same function, by setting this percentage to zero. See [20] for more details about this function.

Number of coefficients equal to zero	Percentage of failures
0%	0%
20%	16%
30%	22%
50%	56%
70%	82%
80%	96%
90%	99%

TABLE 3. Percentage of failures of Algorithm 3.3 against sparseness

The second algorithm is harder to test. Indeed, as we have seen, almost all pairs of dense polynomials will give an affirmative result when put through the first algorithm. Thus we turned to sparse polynomials. Unfortunately, most pairs that fail the first algorithm will also fail the second. Moreover, the few that did not had coefficients so large that SINGULAR had difficulty dealing with them.

We got around this problem by writing a program that generates 1-forms for which the polynomial $R(x)$ (of Step 3 of Algorithm 3.3) is a reducible polynomial in x. More precisely, let

$$a = -yx^n + g(x) \quad \text{and} \quad b = x^{n+1} - f(x,y), \tag{5.2}$$

where $f(x,y)$ is a polynomial of degree n in $\mathbb{Q}[x,y]$ and g a polynomial of degree n in $\mathbb{Q}[x]$. Suppose also that $f(0,y) \neq 0$. If $a = 0$ then $y = g(x)/x^n$. Taking this into the equation $b = 0$, we find that $p = 0$, where

$$p = x^{n^2+n+1} - x^{n^2} f\left(x, \frac{g(x)}{x^n}\right). \tag{5.3}$$

The program generates a reducible polynomial p of degree n^2+n+1 and, by comparing coefficients with (5.3) finds a and b. The resulting foliation, which fails Algorithm 3.3 by construction, is then tested with Algorithm 4.3. The program proceeds as follows:

Algorithm 5.1. *Given n and a partition $2 \leq d_1 \leq d_2 \leq \cdots \leq d_t$ of n^2+n+1, the algorithm returns either a 1-form $adx + bdy$ such that Algorithm 3.3 fails at ω, while Algorithm 4.3 is successful, or* `failure`.

Step 1: *Find distinct primes $p_1, \ldots, p_t$.*
Step 2: *Construct monic polynomials $f_1, \ldots, f_t$, of the form*

$$f_j = x^{d_j} + \sum_{i=1}^{d_j-1} c_i p_j x^i + p_j,$$

where the c_i are integers generated at random.
Step 3: *Let $F = f_1 \cdots f_t$.*
Step 4: *Comparing coefficients, as explained above, find (if they exist) polynomials a and b such that*

$$(a,b) \cap \mathbb{Q}[x] = (F).$$

If a and b cannot be found return `failure`.
Step 5: *Apply Algorithm 4.3 to the foliation defined by $adx + bdy$.*
Step 6: *If the algorithm fails return* `failure`*, otherwise return a and b.*

Two steps of this algorithm need some amplification. First, the polynomials f_j constructed in Step 2 are irreducible by Eisenstein's criterion. Therefore, all the factors of $F = f_1 \cdots f_t$ have multiplicity one.

Second, the algorithm assumes, for the sake of simplicity, that (in the notation of (5.2)) $g = 1$ and f is a dense polynomial of degree n with undetermined coefficients. A simple calculation shows that, in this case, the polynomial p obtained in (5.3) is not dense. Moreover, this would be the case even if we had not assumed that $g = 1$. For this reason we limited our tests to two cases:

- $n = 2$ with the partitions $3 \leq 4$ and $2 \leq 2 \leq 3$ of 7, and
- $n = 3$ with the partitions $3 \leq 4 \leq 6$ and $6 \leq 7$ of 13.

When $n = 2$, we chose

$$F = \begin{cases} (x^3 + c_1 p_1 x^2 + p_2)(x^4 + d_1 p_2 x^3 + d_2 p_2^2 + p_2) & \text{for the partition } 3 \leq 4 \\ (x^2 + p_1)(x^2 + p_2)(x^3 + c_1 p_3 x^2 + p_3) & \text{for the partition } 2 \leq 2 \leq 3, \end{cases}$$

and, when $n = 3$,

$$F = \begin{cases} (x^3 + p_1)(x^4 + p_2)(x^6 + c_1 p_3 x^4 + p_3) & \text{for the partition } 3 \leq 4 \leq 6 \\ (x^6 + c_1 p_1 x^4 + p_1)(x^7 + c_1 p_2 x^6 + c_2 p_2 x^4 + p_4) & \text{for the partition } 6 \leq 7, \end{cases}$$

We ran the algorithm 100 times for each of the partitions listed above, and it never reported a failure. The average time required to construct an example is given in table 4.

Partition	Average time
$3 \leq 4$	125 ms
$2 \leq 2 \leq 3$	500 ms
$3 \leq 4 \leq 6$	1 min
$6 \leq 7$	10 s

TABLE 4. Algorithm 5.1

As the experimental tests show, Algorithm 3.3 will prove that any sufficiently generic 1-form of type (5.1) in $\mathbb{Q}[x, y]$ gives rise to a foliation of $\mathbb{P}^2$ without algebraic solutions. Moreover, the algorithm is very efficient even for foliations of a fairly large degree. Algorithm 4.3, on the other hand, suffers from serious problems caused by coefficient explosion. Indeed it will almost certainly fail to return any result on randomly generated forms for which Algorithm 3.3 fails. Despite that we were able to construct many examples on which the first algorithm fails, while the second successfully detects that the foliation does not have any algebraic solutions.

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