ON SOME FOLIATIONS ARISING IN $\mathcal{D}$-MODULE THEORY

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Abstract. We describe the properties of some foliations which arise in the study of the characteristic variety of $\mathcal{D}$-modules constructed from vector fields of an affine space.

1. Introduction and motivation

In a paper [14] of 1878 G. Darboux proposed a method for finding a first integral of a differential equation in terms of the algebraic curves tangent to the vector field that defines that equation, the *invariant algebraic curves* of the vector field. Darboux also pointed out the importance of studying the singularities of the differential equation to the analysis of the invariant algebraic curves. Darboux's ideas were taken up in the 19th century by Poincaré and have recently flourished in the work of several mathematicians, among them Jouanolou [23], Cerveau and Lins Neto [6], Carnicer [5] and Walcher [31].

Using the language of algebraic geometry we may generalize invariant algebraic curves to higher dimensional varieties. Let X be a smooth complex algebraic variety over which a one dimensional foliation $\mathcal{F}$ has been defined. Such a foliation corresponds to a map $f : \Omega^1_X \to \mathcal{L}$, where Ω^1_X is the sheaf of Kähler differentials and $\mathcal{L}$ is a line bundle over X. Dualizing this sequence and tensoring it up with $\mathcal{L}^{-1}$, we get a homomorphism $\mathcal{O}_X \to \mathcal{L}^{-1} \otimes \Theta_X$. Thus, the foliation $\mathcal{F}$ may also be defined by a section of the sheaf $\mathcal{L}^{-1} \otimes \Theta_X$. A point $x \in X$ is a *singularity* of $\mathcal{F}$ if f is not surjective at x. The set of all singularities of $\mathcal{F}$ will be denoted by $\mathrm{Sing}(\mathcal{F})$. A subscheme Y of X is *invariant* under f if there exists a map $\Omega^1_Y \to \mathcal{L}|_Y$ such that the diagram

$$\begin{array}{ccc} \Omega^1_X|_Y & \xrightarrow{f|_Y} & \mathcal{L}|_Y \\ \downarrow & \nearrow & \\ \Omega^1_Y & & \end{array}$$

is commutative. For more details see [13]. The study of invariant algebraic subvarieties in this more general setting has been considered by Soares [28], [29], Esteves [17], and Esteves and Kleiman [18], [19], among others.

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In this note we study the singularities (section 2) and invariant algebraic subvarieties (section 3) of foliations of $\mathbb{P}^n \times \mathbb{P}^n$ induced by hamiltonian vector fields determined by bihomogeneous polynomial functions of $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ that are linear on the ys. The motivation for studying so special a case comes from the interplay between symplectic geometry and the theory of $\mathcal{D}$-modules.

A $\mathcal{D}$-module is a module over a ring of differential operators of a smooth complex algebraic variety X. Since these rings are noncommutative, their modules often have a very rich structure, which can be studied with the help of a very important geometric invariant, the *characteristic variety*, which is a subvariety of the cotangent bundle T^*X. The cotangent bundle has a natural symplectic structure, relative to which the characteristic variety of a $\mathcal{D}(X)$-module has to be co-isotropic; see [20] or [8] for more details.

The most important special case of this construction is arguably that of the ring of differential operators of the complex affine space $\mathbb{A}^n$. As has been shown in [30], [3], [9], [10] and [15], quotients of these rings by cyclic left ideals generated by operators of order one are an excellent source of examples of $\mathcal{D}(\mathbb{A}^n)$-modules with various interesting properties. It turns out that such modules have for their characteristic varieties hypersurfaces defined by polynomials that are linear in the fibres.

More precisely, if $x_1, \dots, x_n$ are coordinates of $\mathbb{A}^n$ and $y_1, \dots, y_n$ the corresponding conjugate coordinates on the fibres of $T^*\mathbb{A}^n$, then these polynomials can be written in the form $f = \sum_{i=1}^n a_i y_i$, where $a_i \in \mathbb{C}[x_1, \dots, x_n]$ for $1 \leq i \leq n$. In this case, the hamiltonian vector field ξ_f induced by f has the form given in equation (2.1). The co-isotropy implies that the characteristic variety of any submodule or quotient of a module whose characteristic variety has equation $f = 0$ is invariant under ξ_f.

By construction, these characteristic varieties are always *conical*, that is homogeneous with respect to the ys. So, introducing a new variable x_0, we can homogenize both ξ_f and f with respect to the xs. The resulting vector field of $\mathbb{C}^{n+1} \times \mathbb{C}^n$ induces a foliation in $X = \mathbb{P}^n \times \mathbb{P}^{n-1}$, which leaves the hyperplane $x_0 = 0$ invariant. Since this hyperplane is naturally isomorphic to $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$, the foliation that ξ_f induces on it is an example of the kind of foliation we propose to study in this note. This particular foliation plays an important rôle in the solution of a conjecture of Bernstein and Lunts, see [11].

For another interesting example, we turn to conormal varieties. Keeping the notation above for the coordinates, let I be a homogeneous ideal of $\mathbb{C}[x_1, \dots, x_n]$ and consider the conormal variety Y with support on $Z \subset \mathbb{A}^n$, the variety of zeroes of I. In other words, Y is the closure in the cotangent bundle of $\mathbb{A}^n$ of the conormal bundle of $Z \backslash \mathrm{Sing}(Z)$. A polynomial vector field of $\mathbb{A}^n$ gives rise to a regular function on its cotangent bundle. Moreover, if Z is invariant under such a field then the corresponding map f vanishes on Y. This implies that Y is invariant under the hamiltonian vector field ξ_f. Since I is homogeneous with respect to both the xs (by hypotheses) and the ys (by construction), it determines a variety $\overline{Y}$ of $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$. Assuming that f is homogeneous in the xs, it follows that ξ_f induces a foliation $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ that leaves $\overline{Y}$ invariant. For more on conormal varieties and their relevance to the theory of $\mathcal{D}$-modules see [7, 49ff], [24, chapter 2] and [12].

2. Singularities

2.1. **Definitions and notation.** Let n be a positive integer and $\mathbf{a} = (a_0, \dots, a_n)$ be an n-tuple of homogeneous polynomials in $\mathbb{C}[x_0, \dots, x_n]$, all of which have the same degree $\deg(\mathbf{a}) = k \geq 2$. If

$$h_{\mathbf{a}} = \sum_{i=0}^{n} a_i y_i, \quad d_{\mathbf{a}} = \sum_{i=0}^{n} a_i \frac{\partial}{\partial x_i}, \quad \text{and} \quad A_i = \sum_{j=0}^{n} \frac{\partial h_{\mathbf{a}}}{\partial x_i} y_j$$

then the bihomogeneous vector field

$$\xi_{\mathbf{a}} = d_{\mathbf{a}} - \sum_{i=0}^{n} A_i \frac{\partial}{\partial y_i}. \tag{2.1}$$

defines a foliation of $\mathbb{P}^n \times \mathbb{P}^n$, which we denote by $\mathcal{F}_{\mathbf{a}}$, while $\Phi_{\mathbf{a}}$ will stand for the foliation of $\mathbb{P}^n$ induced by $d_{\mathbf{a}}$. We will write π_1 and π_2 for the projections of $\mathbb{P}^n \times \mathbb{P}^n$ on its first and second factors, respectively. Under these hypotheses, $\xi_{\mathbf{a}}$ is a global section of $T_{\mathbf{a}} \otimes_{\mathcal{O}_X} T_X$, where

$$T_{\mathbf{a}} = \mathcal{O}_X(k-1, 0) = \pi_1^*(\mathcal{O}_{\mathbb{P}^n}(k-1)) \otimes_{\mathcal{O}_X} \pi_2^*(\mathcal{O}_{\mathbb{P}^n}),$$

is called the *tangent sheaf* of $\mathcal{F}_{\mathbf{a}}$. This sheaf fits into an exact sequence

$$0 \to T_{\mathbf{a}} \to T_X \to \mathcal{N}_{\mathbf{a}} \to 0,$$

where T_X is the tangent sheaf of X. The cokernel $\mathcal{N}_{\mathbf{a}}$ is called the *normal sheaf* of $\mathcal{F}_{\mathbf{a}}$.

2.2. **Singularities of Poincaré type.** Let $[p] \times [q] \in \mathbb{P}^n \times \mathbb{P}^n$ be a singularity of $\mathcal{F}_{\mathbf{a}}$. It follows from the definition given in section 1 that this is equivalent to saying that $[p] \times [q]$ is a zero of the two by two minors of the matrices

$$\begin{bmatrix} a_0 & \cdots & a_n \\ x_0 & \cdots & x_n \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A_1 & \cdots & A_n \\ y_1 & \cdots & y_n \end{bmatrix}.$$

In particular, $[p] \in \mathbb{P}^n$ is a singularity of $\Phi_{\mathbf{a}}$. We proceed to analyse the singularities of $\mathcal{F}_{\mathbf{a}}$. More precisely, we aim to show that, under certain hypotheses, the $2n$-tuple $\mu = (\mu_1, \dots, \mu_{2n})$ of eigenvalues of the 1-jet of $\mathcal{F}_{\mathbf{a}}$ at a singularity $[p] \times [q]$ satisfies the following conditions:

(1) μ is nonresonant;
(2) none of the ratios of these eigenvalues is real.

Since $GL_{n+1}(\mathbb{C})$ acts transitively in $\mathbb{P}^n$, there exists $g \in GL_{n+1}(\mathbb{C})$ such that $g \cdot [p] = [e_0]$, where $e_i \in \mathbb{C}^{n+1}$ denotes the vector all of whose coordinates are zero, except for the i-coordinate, which is equal to one. Then,

$$G = \begin{bmatrix} g & 0 \\ 0 & (g^{-1})^t \end{bmatrix} \in \mathrm{Sp}(2(n+1), \mathbb{C}),$$

the symplectic group over $\mathbb{C}$. Thus, performing on $\mathcal{F}_{\mathbf{a}}$ the change of coordinates induced by G, we end up with a foliation which is still of the form (2.1). So we may assume that $\mathcal{F}_{\mathbf{a}}$ has a singularity at $p = [e_0]$ We will dehomogenize $\mathcal{F}_{\mathbf{a}}$ in order to study the local behaviour of invariant varieties in the neighbourhood of the singularity $[p] \times [q]$. Without loss of generality, we may also assume that the first coordinate of q is nonzero. Performing the dehomogenization, we obtain the vector field

$$F_{\mathbf{a}} = \sum_{i=1}^{n} (\widehat{a}_i - x_i \widehat{a}_0) \frac{\partial}{\partial x_i} - \sum_{j=1}^{n} \left(\widehat{A}_j - y_j \widehat{A}_0 \right) \frac{\partial}{\partial y_j},$$

where the circumflex means that the corresponding polynomial has been dehomogenized by taking $x_0 = 1$ and $y_0 = 1$. Since $p = [e_0]$ is a singularity of $\Phi_{\mathbf{a}}$, it follows that

$$\widehat{a}_i(0) = (\widehat{a}_i - x_i\widehat{a}_0)(0) = 0.$$

Moreover, $a_i - x_0^{k-1}\ell_i \in (x_1, \dots, x_n)$, for some linear form $\ell_i \in \mathbb{C}[x_0, \dots, x_n]$, so that

$$\ell_i(p) = a_i(p) = \begin{cases} 0 & \text{if} \quad i \geq 1 \\ \ell_0(p) & \text{otherwise.} \end{cases} \tag{2.2}$$

Since the number $\ell_0(p)$ appears quite often in what follows, we denote it by α. Next we must compute the 1-jet of $F_{\mathbf{a}}$ at $[p] \times [q]$. But

$$\frac{\partial a_i}{\partial x_0} - (k-1)x_0^{k-2}\ell_i - x_0^{k-1}\ell_i(p) \in (x_1, \dots, x_n).$$

Thus, taking (2.2) into account,

$$\frac{\partial a_i}{\partial x_0}(p) = 0 \quad \text{for } i > 0, \text{ while} \quad \frac{\partial a_0}{\partial x_0}(p) = k\alpha.$$

Since the first n terms are independent of the ys, the 1-jet can be written in the form

$$\begin{bmatrix} J_1 & 0 \\ * & -J_2 \end{bmatrix}$$

where J_1 and J_2 are $n \times n$ matrices. A straightforward computation shows that the ij entry of J_1 satisfies

$$(J_1)_{ij} = \begin{cases} \frac{\partial \widehat{a}_i}{\partial x_i}(0) - \widehat{a}_0(0) & \text{if } i = j \\ \\ \frac{\partial \widehat{a}_i}{\partial x_j}(0) & \text{otherwise.} \end{cases}$$

Thus, denoting by J_0 the jacobian of $(\widehat{a}_1, \dots, \widehat{a}_n)$, we can write

$$J_1 = J_0 - \widehat{a}_0(0)I_n = J_0 - \alpha I_n$$

where I_n denotes the $n \times n$ identity matrix. Turning now to J_2,

$$(J_2)_{ij} = \begin{cases} \frac{\partial \widehat{a}_j}{\partial x_j}(0) - k\alpha & \text{if } i = j \\ \\ \frac{\partial \widehat{a}_j}{\partial x_i}(0) & \text{otherwise.} \end{cases}$$

Hence,

$$J_2 = J_0^t - k\alpha I_n$$

Therefore the eigenvalues of J are of the form $\lambda - \alpha$ or $-\lambda + k\alpha$ for some eigenvalue λ of J_0. Note that the coefficients of ℓ_i can be chosen arbitrarily, so that both α and the λs can take any value whatsoever.

Writing $\lambda_1, \dots, \lambda_n$ and α to denote the λs and α corresponding to the singularity $[p] \times [q]$, the $2n$-tuple of eigenvalues of the 1-jet of $\mathcal{F}_{\mathbf{a}}$ at $[p] \times [q]$ is

$$\tau(p, q) = (\lambda_1 - \alpha, \dots, \lambda_n - \alpha, -\lambda_1 + k\alpha, \dots, -\lambda_n + k\alpha), \tag{2.3}$$

For the purposes of this note we will say that the singularity $[p] \times [q]$ of $\mathcal{F}_{\mathbf{a}}$ is of *Poincaré type* if

(1) $\lambda_1, \dots, \lambda_n, \alpha$ are linearly independent over $\mathbb{Q}$; and

(2) the ratios

$$\frac{\lambda_i - \nu\alpha}{\lambda_j - \nu'\alpha} \in \mathbb{C} \setminus \mathbb{R}$$

for every $1 \leq i < j \leq n$ and all choices of $\nu, \nu' \in \{1, k\}$.

Proposition 2.1. *If the singularity $[p] \times [q]$ of $\mathcal{F}_{\mathbf{a}}$ is of Poincaré type then the $2n$-tuple $\tau(p,q)$ is nonresonant.*

Proof. Suppose that one of the eigenvalues of $\mathcal{F}_{\mathbf{a}}$ at $[p] \times [q]$, say μ, satisfies a resonance relation. Taking into account our previous characterisation of the eigenvalues of $\mathcal{F}_{\mathbf{a}}$ at a singularity, we conclude that there must exist positive integers q_j and m_j such that

$$\mu = \sum_{i=1}^{n} q_j(\lambda_j - \alpha) + m_j(-\lambda_j + k\alpha).$$

Collecting the terms corresponding to the same λs,

$$\mu = \sum_{i=1}^{n} (q_j - m_j)\lambda_j - \sum_{i=1}^{n} (q_j - km_j)\alpha.$$

Now let

$$e(\mu) = \begin{cases} 0 & \text{if} \quad \mu = \lambda_t - \alpha \\ 1 & \text{if} \quad \mu = -\lambda_t + k\alpha, \end{cases}$$

for some $1 \leq t \leq n$. Since the $\lambda_1, \ldots, \lambda_n, \alpha$ are linearly independent over $\mathbb{Q}$, it follows that,

$$q_j - m_j = \begin{cases} (-1)^{e(\mu)} & \text{if} \quad j = t \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\sum_{i=1}^{n} (q_j - km_j) = \sum_{i=1}^{n} (1-k)m_j + (-1)^{e(\mu)}.$$

Therefore,

$$\sum_{i=1}^{n} m_j = e(\mu) \text{ so that } \sum_{i=1}^{n} m_j + \sum_{i=1}^{n} q_j = 1,$$

no matter what μ is. But this implies that the qs and ms do not define a resonance relation, contrary to what had been assumed. □

Let $\mathcal{X}_{n,k}$ be the projectivization of the complex vector space of n-tuples of homogeneous polynomials of degree k in $\mathbb{C}[x_0, \ldots, x_n]$. We will define two subsets of $\mathcal{X}_{n,k}$ from which foliations are going to be chosen. First, let $\mathcal{P}_{n,k}$ be the subset of those $\mathbf{a} \in \mathcal{X}_{n,k}$ that give rise to a foliation $\mathcal{F}_{\mathbf{a}}$ of $\mathbb{P}^n \times \mathbb{P}^n$ all of whose singularities are of Poincaré type. The second set corresponds to those $\mathbf{a} \in \mathcal{P}_{n,k}$ for which a_0 is irreducible and $(a_0, \ldots, a_n)$ is the irrelevant maximal ideal of $\mathbb{C}[x_1, \ldots, x_n]$. This last set will be denoted by $\mathcal{V}_{n,k}$.

Proposition 2.2. *Both $\mathcal{P}_{n,k}$ and $\mathcal{V}_{n,k}$ are dense subsets of $\mathcal{X}_{n,k}$.*

Proof. The proof that $\mathcal{P}_{n,k}$ is a dense subset of $\mathcal{X}_{n,k}$ is analogous to the proofs of similar results in [25, p.668ff] and [28, Theorem 2, p. 144], and will be omitted. In order to prove that $\mathcal{V}_{n,k}$ is also dense in $\mathcal{X}_{n,k}$ we consider the set

$$\mathcal{U}_{n,k} = \{\mathbf{a} \in \mathcal{X}_{n,k} : a_0 \text{ is irreducible and } \ (a_0, \dots, a_n) = (x_0, \dots, x_n)\}.$$

The density of $\mathcal{V}_{n,k}$ follows from the inclusion

$$\mathcal{P}_{n,k} \cap \mathcal{U}_{n,k} \subset \mathcal{V}_{n,k},$$

if we prove that $\mathcal{U}_{n,k}$ is an open nonempty subset of $\mathcal{X}_{n,k}$. In order to do that, let

$$\mathcal{Y}_{n,k} = \{[\mathbf{a}] \times [p] : \mathbf{a}(p) = 0\} \subset \mathcal{X}_{n,k} \times \mathbb{P}^n.$$

If ϕ is the projection of $\mathcal{X}_{n,k} \times \mathbb{P}^n$ on its first factor then

$$F_{n,k} = \{[\mathbf{a}] \in \mathcal{X}_{n,k} : \dim((\phi|_{\mathcal{Y}_{n,k}})^{-1}([\mathbf{a}])) \geq 0\}$$

is closed in $\mathcal{X}_{n,k}$. Since $[x_0^k : \cdots : x_n^k]$ does not belong to this set, its complement is a nonempty open subset of $\mathcal{X}_{n,k}$. Let S_k be the homogeneous component of degree k of $\mathbb{C}[x_0, \dots, x_n]$. A similar argument shows that

$$G_{n,k} = \{[\mathbf{a}] \in \mathcal{X}_{n,k} : a_0(p) = 0 = \nabla a_0(p) \text{ for some } \ p \in \mathbb{P}^n\} \subset \mathbb{P}(S_k)$$

is also closed in $\mathcal{X}_{n,k}$. Since an n-tuple with $a_0 = x_0^k + \cdots + x_n^k$ does not belong to $G_{n,k}$, it follows that

$$\mathcal{U}_{n,k} = \mathcal{X}_{n,k} \setminus (F_{n,k} \cup G_{n,k})$$

is an open nonempty set of $\mathcal{X}_{n,k}$ as we wished to prove. □

2.3. **Some global properties.** Before we proceed to the local analysis of $\xi_{\mathbf{a}}$, let us consider the effect of Poincaré type singularities on the hypersurface $\mathcal{Z}(h_{\mathbf{a}})$ itself. We retain the notation used in §2.2 and write

$$\nu(n, d) = d^n + d^{n-1} + \cdots + d + 1.$$

It follows from [29, Remark 3.2, p. 498] that this is the number of singularities of a nondegenerate holomorphic foliation of degree d defined in $\mathbb{P}^n$.

Proposition 2.3. *If all the singularities of $\mathcal{F}_{\mathbf{a}}$ are of Poincaré type, then*

(1) $\mathcal{Z}(h_{\mathbf{a}})$ *is smooth;*
(2) $\mathrm{Sing}(\mathcal{F}_{\mathbf{a}}) \not\subset \mathcal{Z}(h_{\mathbf{a}})$*;*
(3) $\#\mathrm{Sing}(\mathcal{F}_{\mathbf{a}}) = (n+1)\nu(n, d)$*;*

where, $d = \deg(\Phi_{\mathbf{a}}) = \deg(a_i) - 1$ *for every* $0 \leq i \leq n$.

Proof. In this proof we use the notation and results established in §2.2. We need only argue over what happens at one singularity of $\mathcal{F}_{\mathbf{a}}$, which can always be taken to be of the form $[e_0] \times [q]$. Moreover, since we are assuming the singularities to be of Poincaré type, there exists a linear change of variables in the xs such that the matrix J_0 is diagonal with eigenvalues $\lambda_1, \dots, \lambda_n$. In these coordinates, we have that

$$\frac{\partial a_i}{\partial x_j}(e_0) = \lambda_i \delta_{ij},$$

for $1 \leq i, j \leq n$, where δ_{ij} is Kronecker's delta symbol. This linear change of the x-coordinates gives rise to a symplectic change of coordinates in $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$. We proceed by explicitly computing these singularities.

If $[e_0] \times [q]$ is a singularity of $\mathcal{F}_{\mathbf{a}}$, then the coordinates of $[q]$ are zeroes of the equations

$$y_j A_i(p, y) - y_i A_j(p, y) = 0 \quad \text{for all} \quad 0 \le i, j \le n; \tag{2.4}$$

which, when $j = 0$, become

$$0 = y_0 A_i(p, y) - y_i A_0(p, y) = y_0 y_i(\lambda_i - k\alpha),$$

by our choice of the x-coordinates. In particular, if $y_0 \neq 0$ then $y_i = 0$ for all $1 \le i \le n$; so $[q] = [e_0]$ is a solution of the system (2.4). If, on the other hand, $y_0 = 0$, then the equations we have to consider are

$$0 = y_j A_i(p, y) - y_i A_j(p, y) = y_j y_i(\lambda_i - \lambda_j)$$

for all $1 \le i < j \le n$, whose zeroes are the points $[e_i]$, for $1 \le i \le n$. Therefore, $[e_0] \times [q]$ is a singularity of $\mathcal{F}_{\mathbf{a}}$ if and only if $[q] = [e_j]$ for $0 \le j \le n$.

Turning now to (1), let us consider a singularity of $\mathcal{Z}(h_{\mathbf{a}})$. Since the change of coordinates used above is symplectic, we may assume that this singularity is of the form $[e_0] \times [q]$, for some $[q] \in \mathbb{P}^n$. Thus, $[q]$ must be a zero of

$$\frac{\partial h_{\mathbf{a}}}{\partial y_j}(e_0) = a_j(e_0) = 0 \quad \text{and} \quad \frac{\partial h_{\mathbf{a}}}{\partial x_j}(e_0) = A_j(e_0, q) = 0, \tag{2.5}$$

for all $0 \le j \le n$. In particular, $[e_0] \times [q]$ must be a singularity of $\mathcal{F}_{\mathbf{a}}$, which is of Poincaré type by hypothesis. This implies that $\alpha \neq 0$ by (2.2). However, by (2.3), $a_0(e_0) = \alpha$, which contradicts (2.5) when $j = 0$.

In order to prove (2), recall that if $[e_0]$ is a singularity of $\Phi_{\mathbf{a}}$, then $\Phi_{\mathbf{a}}(e_0)$ is collinear with e_0 as vectors in $\mathbb{C}^{n+1}$. Thus,

$$h_{\mathbf{a}}([e_0] \times [e_j]) = a_j(e_0) = \delta_{j0}\alpha;$$

so $\mathcal{Z}(h_{\mathbf{a}})$ contains all but one of the singularities of $\mathcal{F}_{\mathbf{a}}$ in $\pi_2^{-1}([e_0])$.

Finally, Poincaré type implies that the singularities of $\Phi_{\mathbf{a}}$ in $\mathbb{P}^n$ are all nondegenerate. Therefore, by [29, Remark 3.2, p. 498], $\Phi_{\mathbf{a}}$ has $d^n + d^{n-1} + \cdots + d + 1$ singularities, where d is the degree of $\Phi_{\mathbf{a}}$. But the above argument implies that, for each $[p] \in \mathrm{Sing}(\Phi_{\mathbf{a}})$ there are exactly $n+1$ points $[q] \in \mathbb{P}^n$ such that $[p] \times [q]$ are singularities of $\mathcal{F}_{\mathbf{a}}$, and this proves (3). □

Note that we could easily have computed the number of singularities of $\mathcal{F}_{\mathbf{a}}$ using the Baum-Bott Theorem; see [2]. However, this seemed pointless, since we also needed to determine the coordinates of the singular points, and that immediately gives a method to count the singularities.

2.4. **Germs of invariant subvarieties.** We now turn to the behaviour of certain invariant subvarieties in the neighbourhood of a Poincaré type singularity p of $\xi_{\mathbf{a}}$. We may choose local coordinates such that $p = (0, \dots, 0)$. Let D be a derivation that represents $\xi_{\mathbf{a}}$ in this neighbourhood. Keeping to the notation used in 2.2, the eigenvalues of the 1-jet of D at p can be written in the form

$$\lambda_1 - \alpha, \dots, \lambda_n - \alpha, -\lambda_1 + k\alpha, \cdots - \lambda_n + k\alpha,$$

where $k \ge 2$ is an integer. Moreover, since p is of Poincaré type, $\alpha, \lambda_1, \dots, \lambda_n$ are linearly independent over $\mathbb{Q}$. In particular, the eigenvalues are nonresonant by

Proposition 2.1. Thus, by Poincaré's Theorem [1, p. 175], D can be written, after a formal change of variables, in the form

$$D = \sum_{i=1}^{n} (\lambda_i - \alpha) x_i \frac{\partial}{\partial x_i} - (\lambda_i - k\alpha) y_i \frac{\partial}{\partial y_i}, \tag{2.6}$$

where $x_1, \dots, x_n, y_1, \dots, y_n$ is a formal coordinates system at p. For the remainder of this subsection we assume that D is of the form (2.6). We need a technical lemma concerning the eigenvectors of D.

Lemma 2.4. *Let p and D be as above. The monomials $x^\beta y^\gamma$ and $x^{\beta'} y^{\gamma'}$ belong to the same eigenspace of D if and only if*

$$\beta - \beta' = \gamma - \gamma', \quad |\beta| = |\beta'| \ \textit{and} \ \ |\gamma| = |\gamma'|,$$

where $|u|$ denotes the sum of the entries of the integer vector u. In particular, an eigenspace that contains a pure power of a variable is one dimensional.

Proof. Let λ denote the vector $(\lambda_1, \dots, \lambda_n)$. Then, $x^\beta y^\gamma$ is an eigenvector of D for the eigenvalue

$$\langle \lambda, \beta - \gamma \rangle + \alpha |k\gamma - \beta|.$$

Since $\lambda_1, \dots, \lambda_n, \alpha$ are linearly independent over $\mathbb{Q}$, the monomials $x^\beta y^\gamma$ and $x^{\beta'} y^{\gamma'}$ have the same eigenvalue if and only if

$$\beta - \gamma = \beta' - \gamma' \ \text{ and } \ |k\gamma - \beta| = |k\gamma' - \beta'|, \tag{2.7}$$

which are equivalent to the conditions stated in the lemma. Thus a monomial $x^{\beta'} y^{\gamma'}$ has the same eigenvalue as x_i^r if and only if

$$re_i = \beta' - \gamma' \ \text{ and } \ -r = |k\gamma' - \beta'|.$$

The first equation implies that

$$\beta'_j = \gamma'_j = 0 \ \text{ for all } \ j \neq i.$$

Taking this into account the two equations above become

$$r = \beta'_i - \gamma'_i \ \text{ and } \ -r = k\gamma'_i - \beta'_i.$$

Since $k > 1$, it follows that $\beta'_i = r$ and $\gamma'_i = 0$ as we wished to prove. The proof for the power y_i^r is analogous and will be omitted. □

The next result is a variant of [26, §2.4, Lemma, p. 543] that has been adapted to the needs of this paper.

Proposition 2.5. *Let $\mathbf{a} \in \mathcal{P}_{n,k}$ and assume that Y and Z are subvarieties of $\mathbb{P}^n \times \mathbb{P}^n$ invariant under $\mathcal{F}_{\mathbf{a}}$. If $\dim(Y) + \dim(Z) = 2n$ and Z is smooth at a point of $Y \cap Z$, then Y is also smooth at this point.*

Proof. For the sake of simplicity we will suppose that $\mathbf{a} \in \mathcal{P}_{n,k}$ has been fixed and drop it from the notation. Now let

$$p \in Y \cap Z \subseteq \operatorname{Sing}(\mathcal{F}).$$

If Z is smooth at p, we may choose local coordinates $z_1, \dots, z_{2n}$ of $\mathbb{P}^n \times \mathbb{P}^n$ at p such that Z is given by $z_1 = \cdots = z_r = 0$ in these coordinates. Let I be the ideal of Y in the local ring $\mathcal{O}_p$ of p and let $J = (z_1, \dots, z_r)$ and $\widehat{I} = \widehat{\mathcal{O}}_p I$, be the ideals of Z and Y in the completion $\widehat{\mathcal{O}}_p$. Denote by D the derivation that $\mathcal{F}$ defines on $\widehat{\mathcal{O}}_p$.

Since J is invariant under D, it induces a derivation $\overline{D}$ in

$$\mathbb{C}[[z_1, \ldots, z_{2n}]]/J \cong \mathbb{C}[[z_{r+1}, \ldots, z_{2n}]].$$

Applying Poincaré's Theorem to $\overline{D}$, we can assume, without loss of generality, that there are distinct eigenvalues $\mu_{r+1}, \ldots, \mu_{2n}$ of the 1-jet of D at p, such that

$$\overline{D} = \sum_{i=r+1}^{2n} \mu_i z_i \partial/\partial z_i.$$

Since $(\widehat{I} + J)/J$ is zero dimensional, it contains a polynomial in $\mathbb{C}[z_i]$, for each $r+1 \leq i \leq 2n$. These polynomials are mapped inside $(\widehat{I} + J)/J$ by $\overline{D}$ and each power of z_i is an eigenvector of $\overline{D}$ associated to a different eigenvalue. Therefore, each one of these powers belong to $(\widehat{I} + J)/J$. Thus, if $r+1 \leq i \leq 2n$, there exists a polynomial $g_i \in \widehat{I}$ of the form

$$g_i = z_i^\ell + \sum_{j=1}^{r} a_{ij} z_j \in \widehat{I}$$

for some integer $\ell > 0$ and $a_{i1}, \ldots, a_{ir} \in \widehat{\mathcal{O}}_p$. Writing g_i as a sum of components, each one of which is a polynomial in a different eigenspace of $\widehat{\mathcal{O}}_p$ under D, and taking into account that $D(g_i) \in \widehat{I}$, the usual argument shows that each one of these components belongs to $\widehat{I}$. However, by Lemma 2.4, no monomials in the support of $\sum_{j=1}^{r} a_{ij} z_j$ can have the same eigenvalue as z_i^ℓ because $i \notin \{1, \ldots, r\}$. Therefore, $z_i^\ell \in \widehat{I}$. Moreover, since I is radical, so is $\widehat{I}$; see [21, Scholie 7.8.3 (vii), p. 215]. Hence, $(z_{r+1}, \ldots, z_n) \in \widehat{I}$. Thus, Y is smooth at p, and the proof is complete. □

3. Invariant subvarieties

In this section we discuss various properties of subvarieties invariant under $\mathcal{F}_{\mathbf{a}}$. We assume throughout the section that

$$h_{\mathbf{a}} = a_0 y_0 + \cdots + a_n y_n \in \mathbb{C}[x_0, \ldots, x_n, y_0, \ldots, y_n]$$

is a bihomogeneous irreducible polynomial. Moreover, $h_{\mathbf{a}}$ will be called *well-chosen* if for some triple $(i, j, k) \in \mathbb{N}^3$ of pairwise distinct integers,

- (a_i) and (a_i, a_j) are prime ideals;
- $a_j \notin (a_i)$ and $a_k \notin (a_i, a_j)$;
- $(a_0, \ldots, a_n) = (x_0, \ldots, x_n)$.

By permuting the y variables, we can always assume that $i = 0$, $j = 1$ and $k = 2$ in the above definition. Indeed, from now on we assume that such a permutation has been performed whenever necessary.

3.1. **Complete intersections.**

We begin with an elementary result in commutative algebra.

Lemma 3.1. *If $h_{\mathbf{a}}$ is well-chosen then the ideals $(h_{\mathbf{a}})$ and $(a_0, h_{\mathbf{a}})$ are both prime in $\mathbb{C}[x_0, \ldots, x_n, y_0, \ldots, y_n]$.*

Proof. Since $h_{\mathbf{a}}$ is linear in the ys, it follows that it can only be factored in the form $b\lambda$, where $b \in \mathbb{C}[x_0, \dots, x_n]$ and λ is linear in the ys. However, this would imply that b is a common divisor of the as, which is not possible because a_0 is irreducible.

We now turn to the ideal $(a_0, h_{\mathbf{a}})$. Let π be the restriction to $\mathcal{Z}(a_0, h_{\mathbf{a}})$ of the projection of $\mathbb{P}^n \times \mathbb{P}^n$ on its first factor. Since a_0 is irreducible, so is the hypersurface $\mathcal{Z}(a_0)$ of $\mathbb{P}^n$. Moreover, the fibre

$$\pi^{-1}([p]) = \mathcal{Z}(h_{\mathbf{a}}(p, y)) \subset \mathbb{P}^n,$$

is linear, therefore irreducible for all $[p] \in \mathcal{Z}(a_0)$. So the fibres have dimension $n-1$ whenever $h_{\mathbf{a}}(p, y) \neq 0$. But $h_{\mathbf{a}}(p, y) = 0$ as a polynomial in $\mathbb{C}[y_0, \dots, y_n]$ if and only if $a_j(p) = 0$ for all $0 \leq j \leq n$. This implies, by the Projective NullstellenSatz, that $(a_0, \dots, a_n) \neq (x_0, \dots, x_n)$, contradicting the hypotheses. Thus $\mathcal{Z}(a_0, h_{\mathbf{a}})$ is an irreducible variety by [22, Theorem 11.14, p. 139].

In order to finish the proof we need only prove that $(a_0, h_{\mathbf{a}})$ is a radical ideal. Suppose that $g^m \in (a_0, h_{\mathbf{a}})$ for some $g \in \mathbb{C}[x_0, \dots, x_n, y_0, \dots, y_n]$ and some integer $m > 0$. Since $\mathbb{C}[x_0, \dots, x_n]/(a_0)$ is a domain, there exists an integer $\ell \geq 0$ and polynomials $q, r \in \mathbb{C}[x_0 \dots, x_n, , y_0 \dots, y_n]$ such that

$$a_1^\ell g \equiv q h_{\mathbf{a}} + r \pmod{(a_0)}$$

and the polynomial r does not contain any monomial with a positive power of y_1. Thus,

$$r^m \equiv (a_1^\ell g)^m \pmod{(a_0, h_{\mathbf{a}})}.$$

Hence, $g^m \in (a_0, h_{\mathbf{a}})$ implies that

$$r^m \equiv b h_{\mathbf{a}} \pmod{(a_0)}, \tag{3.1}$$

for some polynomial b. If

$$b \equiv b_s y_1^s + \cdots + b_0 \pmod{(a_0)},$$

then it follows from (3.1) and the choice of r that $b_s a_1 \equiv 0 \pmod{a_0}$. This implies that $b \equiv 0 \pmod{a_0}$, so $r^m \equiv 0 \pmod{a_0}$. Since (a_0) is a prime ideal, it follows that $r \equiv 0 \pmod{a_0}$ and that

$$a_1^\ell g \equiv q h_{\mathbf{a}} \pmod{(a_0)}. \tag{3.2}$$

Moreover, we may choose the smallest ℓ for which this last congruence holds. If $\ell = 0$ we are done; so let $\ell > 0$ and let us aim at a contradiction.

The minimality of ℓ implies that there exists at least one coefficient in q that does not belong to (a_1, a_0); otherwise we could cancel a_1 in (3.2). Order the monomials in y lexicographically subject to $y_2 > y_3 > \cdots > y_n$ and let y^α be the largest monomial in the support of q whose coefficient q_α does not belong to (a_0, a_1). It follows from (3.2) that $q_\alpha a_2 \in (a_0, a_1)$. Since $h_{\mathbf{a}}$ is well-chosen, it follows that $q_\alpha \in (a_0, a_1)$; which contradicts our choice of α and completes the proof of the theorem. □

Proposition 3.2. *If $h_{\mathbf{a}}$ is well-chosen then all irreducible subvarieties of codimension one in $\mathcal{Z}(h_{\mathbf{a}})$ are schematic complete intersections.*

Proof. Since $h_{\mathbf{a}}$ is well-chosen and has degree one, it follows that $(h_{\mathbf{a}})$ is prime in $\mathbb{C}[x_0, \dots, x_n, y_0, \dots y_n]$. In particular, $B = \mathbb{C}[x_0, \dots, x_n, y_0, \dots y_n]/(h_{\mathbf{a}})$ is a domain. But by Nagata's factoriality lemma, if B_{a_0} is a factorial domain and a_0 is prime in B then B is factorial; see [27, Théoréme 5, p. 31] or [16, Lemma 19.20, p. 487]. However,

$$B_{a_0} \cong \mathbb{C}[x_0, \dots, x_n]_{a_0}[y_1, \dots, y_n]$$

is a factorial domain by [27, Théoréme 4, p. 29 and Corollaire 1, p. 23] and $(a_0, h_{\mathbf{a}})$ is a prime ideal by Theorem 3.1. Therefore,

$$B/(a_0) \cong \mathbb{C}[x_0, \dots, x_n, y_0, \dots y_n]/(a_0, h_{\mathbf{a}})$$

is a domain. Hence, (a_0) is prime in B and, by the factoriality lemma, B is a factorial domain, from which the desired result follows. □

3.2. **Invariant hypersurfaces.** Throughout this subsection we assume that $h_{\mathbf{a}}$ is well-chosen, as defined at the very beginning of this section.

Theorem 3.3. *Let Y be a subvariety of codimension one in $\mathcal{Z}(h_{\mathbf{a}})$ invariant under $\mathcal{F}_{\mathbf{a}}$. If all the singularities of Y are normal crossings then $Y = \mathcal{Z}(h_{\mathbf{a}}, g)$ for some bihomogeneous polynomial g whose bidegree (ℓ, ℓ') satisfies $\ell \le n$ or $\ell' \le n$.*

Proof. In order to simplify the notation write $Z = \mathcal{Z}(h_{\mathbf{a}})$ for the subvariety, $X = \mathbb{P}^n \times \mathbb{P}^n$ for the multiprojective space and $\mathcal{F}$ for the foliation $\mathcal{F}_{\mathbf{a}}$. The ideal sheaf of Z in $\mathcal{O}_X$ is $\mathcal{I}_Z \cong \mathcal{O}_X(-k, -1)$, because $h_{\mathbf{a}}$ has bidegree $(k, 1)$. Since $\omega_X \cong \mathcal{O}_X(-n-1, -n-1)$, it follows that

$$\omega_Z \cong \omega_X \otimes \mathcal{I}_Z \otimes \mathcal{O}_Z \cong \mathcal{O}_Z(k-n-1, -n).$$

However, by adjunction,

$$(\det \mathcal{N}_{\mathbf{a}})^\vee \otimes T_{\mathbf{a}} \cong \omega_Z,$$

where $\mathcal{N}_{\mathbf{a}}$ is the normal bundle of $\mathcal{F} = \mathcal{F}_{\mathbf{a}}$ and $T_{\mathbf{a}}$ its tangent bundle; see § 2.1. Hence,

$$(\det \mathcal{N}_{\mathbf{a}})^\vee \cong \mathcal{O}_Z(-n, -n).$$

By Proposition 3.2, there exists a polynomial $g \in \mathbb{C}[x_0, \dots, x_n, y_0, \dots, y_n]$ of bidegree (ℓ, ℓ') such that $Y = \mathcal{Z}(h_{\mathbf{a}}, g)$. Then $\mathcal{I}_Y \cong \mathcal{O}_Z(\ell, \ell')$. But by [4, equation (3), p. 600] Y can only be invariant under $\mathcal{F}$ if

$$\mathcal{I}_Y \otimes (\det \mathcal{N}_{\mathbf{a}})^\vee \cong \mathcal{O}_Z(\ell - n, \ell' - m)$$

is not ample, which implies that $\ell - n \le 0$ or $\ell' - n \le 0$, and completes the proof of the theorem. □

3.3. **Invariant curves.** We now turn to invariant curves. Recall that a curve C in $\mathbb{P}^n \times \mathbb{P}^n$ has bidegree (a, b) if its class in the Chow ring $A^*(\mathbb{P}^n \times \mathbb{P}^n)$ is $as^nt^{n-1} + bs^{n-1}t^n$, where s and t are the generators of $A^*(\mathbb{P}^n \times \mathbb{P}^n)$. Recall that $\nu(n, k) = k^n + \dots + k + 1$ is the number of singularities of a nondegenerate foliation of degree k in $\mathbb{P}^n$.

Theorem 3.4. *Let $\mathbf{a} \in \mathcal{P}(n, k)$ for some $k \ge 2$. If $C \not\subset \mathcal{Z}(h_{\mathbf{a}})$ is a curve of bidegree (a, b) that is invariant under $\mathcal{F}_{\mathbf{a}}$, then C is smooth and $a + kb \le (n+1)\nu(n, k)$.*

Proof. It follows from Proposition 2.5 that C is smooth and transversal to the hypersurface $Z = \mathcal{Z}(h_{\mathbf{a}})$. The class of C in the Chow ring of X is $[C] = as^nt^{n-1} + bs^{n-1}t^n$ while the class of $Z = \mathcal{Z}(h)$ is $[Z] = ks + t$. Thus,

$$[C] \cdot [Z] = (a + kb). \tag{3.3}$$

Since the intersection is transversal, we also have that $[C] \cdot [Z] = \#(C \cap Z)$. But if two invariant varieties intersect at an isolated point, it must be a singularity of the foliation, so

$$\#(C \cap Z) \leq \#(\mathrm{Sing}(\mathcal{F}_{\mathbf{a}}) \cap Z) \leq (n+1)\nu(n,k), \tag{3.4}$$

where the last inequality comes from Proposition 2.3. The inequality of the theorem follows by combining (3.3) and (3.4). □

When $n = 2$ we can also determine some bounds for invariant curves that are contained in the hypersurface $\mathcal{Z}(h_{\mathbf{a}})$. The first results we prove are concerned with the variety $\mathcal{V}_{\mathbf{a}} = \mathcal{Z}(h_{\mathbf{a}}, \Delta)$ where $\Delta = x_0y_0 + x_1y_1 + x_2y_2$.

Proposition 3.5. *The variety $\mathcal{V}_{\mathbf{a}}$ is isomorphic to the blowup of $\mathbb{P}^2$ at the singular points of $d_{\mathbf{a}}$.*

Proof. Let π be the projection of $\mathbb{P}^2 \times \mathbb{P}^2$ on its first coordinate and denote by $\mathcal{B}$ the blowup of $\mathbb{P}^2$ at the singularities of $d_{\mathbf{a}}$. We will prove that for every $0 \leq i \leq 2$ there exists an isomorphism

$$\phi_i : \mathcal{V}_{\mathbf{a}} \cap \pi^{-1}(U_i) \to \mathcal{B} \cap \pi^{-1}(U_i),$$

where U_i is the open set of $\mathbb{P}^2$ defined by $x_i \neq 0$. The proposition follows from the compatibility of these isomorphisms at the intersections $U_i \cap U_j$, for $0 \leq i < j \leq 2$.

By symmetry it is enough to describe the construction of ϕ_i when $i = 0$. Denoting by $\widehat{f}$ the polynomial obtained by setting $x_0 = 1$ in $f \in \mathbb{C}[x_0, x_1, x_2, y_0, y_1, y_2]$, we can identify $\mathcal{B} \cap \pi^{-1}(U_i)$ with the subset

$$\mathcal{B}_0 = \{(x_1, x_2) \times [y_1 : y_2] \,|\, (\widehat{a}_1 - x_1\widehat{a}_0)y_1 + (\widehat{a}_2 - x_2\widehat{a}_0)y_2 = 0\}$$

of $\mathbb{C}^2 \times \mathbb{P}^1$, and $\mathcal{V}_{\mathbf{a}} \cap \pi^{-1}(U_i)$ with the subset

$$\mathcal{V}_0 = \{(x_1, x_2) \times [y_0 : y_1 : y_2] \,|\, \widehat{h}_{\mathbf{a}} = \widehat{\Delta} = 0\},$$

of $\mathbb{C}^2 \times \mathbb{P}^2$. Under these identifications define

$$\phi_0 : \mathcal{V}_0 \to \mathbb{C}^2 \times \mathbb{P}^2,$$

by

$$\phi_0((x_1, x_2) \times [y_0 : y_1 : y_2]) = (x_1, x_2) \times [y_1 : y_2].$$

That this map is well defined follows from

$$0 = h_{\mathbf{a}}(1, x_1, x_2, y_0, 0, 0) = y_0.$$

Moreover, since

$$\sum_{i=1}^{2} (\widehat{a_i} - x_i\widehat{a_0})y_i = \widehat{h}_{\mathbf{a}} - a_0\widehat{\Delta} = 0;$$

the image of ϕ_0 is contained in $\mathcal{B}_0$. A similar argument allows us to define the inverse of ϕ_0 by

$$(\phi_0)^{-1}((x_1, x_2) \times [y_1 : y_2]) = (x_1, x_2) \times [-x_1y_1 - x_2y_2 : y_1 : y_2].$$

The compatibility of the isomorphisms ϕ_j follows from a simple calculation that will be left to the reader. □

As a consequence of this proposition we can immediately determine a number of properties of $\mathcal{V}_{\mathbf{a}}$.

Corollary 3.6. *$\mathcal{V}_{\mathbf{a}}$ is a smooth variety of dimension 2 birational do $\mathbb{P}^2$, whose Picard group is*

$$\mathbb{Z}h + \sum_{i=1}^{\nu(n,k)} \mathbb{Z}E_i,$$

where h denotes the hyperplane class and E_i, for $1 \le i \le \nu(n,k)$, are the classes of the exceptional fibres.

For the remainder of this note we assume that $\mathbf{a} \in \mathcal{P}_{2,k}$, for some integer $k \ge 2$. Recall that π_1 denotes the projection of $\mathbb{P}^n \times \mathbb{P}^n$ on its first factor and $\Phi_{\mathbf{a}}$ is the foliation induced by $\mathcal{F}_{\mathbf{a}}$ on $\mathbb{P}^n$.

Theorem 3.7. *Let C be an irreducible curve contained in $\mathcal{V}_{\mathbf{a}}$ and invariant under $\mathcal{F}_{\mathbf{a}}$. If $\dim(\pi_1(C)) = 1$ then C is smooth and its class in $\operatorname{Pic}(\mathcal{V}_{\mathbf{a}})$ is*

$$dH + \sum_{i=1}^{\nu(n,k)} \ell_i E_i$$

where $0 \le d \le k+2$ and $0 \le \ell_i \le 2$.

Proof. The projection $\pi_1(C)$ must be a curve invariant under the foliation $\Phi_{\mathbf{a}}$ of $\mathbb{P}^2$, and the hypotheses on $\mathbf{a}$ imply that the singularities of this curve must be normal crossings; see [25, Proposition 2.5, p. 656]. Therefore, by [6, Remark, p. 891], the degree of $\pi_1(C)$ cannot exceed $k+2$. Thus C, which is contained in the strict transform of $\pi_1(C)$, must be smooth. Moreover, the normal crossing condition implies that $\pi_1(C)$ has at most two branches at each of the singularities of $d_{\mathbf{a}}$. Therefore, $0 \le \ell_i \le 2$ and the proof is complete. □

In order to apply this result to an invariant hypersurface of $\mathcal{Z}(h_{\mathbf{a}})$ we need an additional proposition.

Proposition 3.8. *Let g be a bihomogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2, y_0, y_1, y_2]$. If $\mathcal{Z}(h_{\mathbf{a}}, g)$ is invariant under $\xi_{\mathbf{a}}$ then*

$$\dim(\pi_1(\mathcal{Z}(h_{\mathbf{a}}, g, \Delta)) > 0.$$

Proof. One easily checks that $\xi_{\mathbf{a}}(\Delta) = h_{\mathbf{a}}$, which implies that $(h_{\mathbf{a}}, g, \Delta)$ is invariant under $\xi_{\mathbf{a}}$. Let $Y = \pi_1(\mathcal{Z}(h_{\mathbf{a}}, g, \Delta))$ and suppose, by contradiction, that it has dimension zero. Since Δ is invariant under the linear changes of variable described in 2.2, we can assume, without loss of generality that $Y \cap \mathcal{Z}(x_i) = \emptyset$ for every $1 \le i \le n$. But Y must be invariant under $\Phi_{\mathbf{a}}$, so it is a union of singularities of this foliation. Thus, the ideal of Y in $\mathbb{C}[x_0, x_1, x_2, y_0, y_1, y_2]$ is $\mathrm{I}(Y) = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_t$, where the $\mathfrak{m}$s are homogeneous maximal ideals corresponding to singularities of $\Phi_{\mathbf{a}}$ in Y. Hence, there exists a positive integer m such that $(y_i\mu)^m \in (f, \Delta, g)$, for every generator μ of $\mathrm{I}(Y)$. Taking $y_j = 0$ for every $j \neq i$ and $y_i = 1$, we conclude that $\mu \in \sqrt{(a_i, x_i, \beta_i)}$ where β_i is the coefficient of y_i^r in g. Hence, $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_t \subset \sqrt{(a_i, x_i, \beta_i)}$. But x_i does not divide a_i, so $\mathcal{Z}(a_i, x_i, \beta_i) \subset \mathbb{P}^2$ cannot have positive dimension. Hence, renumbering the $\mathfrak{m}$s, if necessary, we have an equality $\sqrt{(a_i, x_i, \beta_i)} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_s$. In particular, this implies that $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_s$ contains a power of x_i, which contradicts our choice of coordinates. □

The following corollary is an immediate consequence of Theorem 3.7 and Proposition 3.8.

Corollary 3.9. *Let C be a curve, contained in $\mathcal{Z}(h_{\mathbf{a}})$, that is invariant under $\xi_{\mathbf{a}}$. If the ideal generated by $\mathrm{I}(C)$ and Δ is a radical ideal, then C is the scheme theoretic intersection of $\mathcal{Z}(h_{\mathbf{a}})$ and $\mathcal{Z}(g)$, where g is a bihomogeneous polynomial of bidegree (d, ℓ) with $d \leq k+2$ and $0 \leq \ell \leq 2$.*

We can also use these results to give a simpler proof of Lemma 5.1 of [11].

Corollary 3.10. *If $\Phi_{\mathbf{a}}$ has no invariant curve in $\mathbb{P}^2$, then*

(1) $\mathcal{Z}(h_{\mathbf{a}}, g, \Delta) = \mathcal{Z}(h_{\mathbf{a}}, g)$;
(2) $g \equiv \Delta^m \pmod{h_{\mathbf{a}}}$ *for some positive integer m.*

Proof. Since the projection of $\mathcal{Z}(h_{\mathbf{a}}, g, \Delta)$ on $\mathbb{P}^2$ is invariant under $\Phi_{\mathbf{a}}$, the hypotheses on $h_{\mathbf{a}}$ imply that it must have dimension zero or two. Dimension zero is excluded by Proposition 3.8; so $\dim \pi_1(\mathcal{Z}(h_{\mathbf{a}}, g, \Delta)) = 2$, which implies that $\mathcal{Z}(h_{\mathbf{a}}, g, \Delta)$ itself has dimension two. Thus, $\Delta^m \in (h_{\mathbf{a}}, g)$ for some positive integer m. □

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