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CONORMAL VARIETIES AND CHARACTERISTIC VARIETIES

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ABSTRACT. We show that the conormal variety of a quasihomogeneous hypersurface in $\mathbb{C}^n$, for $n \geq 4$, whose link is a $\mathbb{Q}$-homology sphere is not the characteristic variety of any $\mathcal{D}$-module.

1. INTRODUCTION

Let X be a complex analytic manifold and $\mathcal{D}_X$ its sheaf of differential operators. The most important geometric invariant of a $\mathcal{D}_X$-module is its characteristic variety. This is a conical subvariety of the cotangent bundle $\mathrm{T}^\star X$. It is well-known that the cotangent bundle is a symplectic manifold and that the characteristic variety of a $\mathcal{D}_X$-module is involutive with respect to the symplectic structure. Thus, there are two conditions that a subvariety of $\mathrm{T}^\star X$ must satisfy in order to be a characteristic variety: it must be conical and involutive.

Conormal varieties are involutive and conical. In fact they are exactly the lagrangian conical subvarieties of $\mathrm{T}^\star X$ (see, for example, [Ge, Proposition 3.1, p. 29]). Furthermore, an irreducible lagrangian variety is always a component of the characteristic variety of some $\mathcal{D}_X$-module (see [CL]).

As pointed out by Malgrange in [Ma2, p. 9], no necessary and sufficient conditions are known for a variety to be a characteristic variety. However, an unpublished result of Kashiwara states that the conormal variety of a quadratic cone in $\mathbb{C}^n$, for $n \geq 5$ odd, is not a characteristic variety (see [CL] for a proof). In this paper we extend Kashiwara's result with the following theorem.

1.1 Theorem. *Let $n \geq 4$ and let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a quasihomogeneous polynomial. If the hypersurface $\mathcal{Z}(f)$ has only an isolated singularity at the origin and its link is a $\mathbb{Q}$-homology sphere, then its conormal variety is not the characteristic variety of any $\mathcal{D}_{\mathbb{C}^n}$-module.*

A simple specific example of a polynomial that satisfies these hypotheses is $f = x_1^{a_1} + \dots + x_n^{a_n}$ with $n \geq 4$, and $a_1, \dots, a_n$ pairwise coprime positive integers. See §4 for more details.

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2. Preliminaries

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial. We say that f is *non-degenerate* if $\mathcal{Z}(f)$ is singular but has only an isolated singularity at the origin. We call f *quasihomogeneous* if there exist positive integers $w_1, \ldots, w_n$ such that f is homogeneous when we set $\deg x_i = w_i$. Equivalently $rf = \sum_{i=1}^n w_i x_i \partial f/\partial x_i$, for some positive integer r, the weighted degree of f.

Notation. Throughout the rest of the paper let $n \geq 4$ and let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-degenerate quasihomogeneous polynomial of weighted degree r with respect to the weights $w_1, \ldots, w_n$. We will denote the corresponding Euler vector field by $E = \sum_{i=1}^n w_i x_i \partial/\partial x_i$.

Let V be the space $\mathbb{C}^n$ with its analytic topology and C the hypersurface $\mathcal{Z}(f)$. Note that $C' = C \setminus \{0\}$ is a smooth divisor on $V' = V \setminus \{0\}$. The *conormal variety* of C, denoted $\mathrm{T}^\star_C V$, is the closure in $\mathrm{T}^\star V$ of the conormal bundle of C' in $\mathrm{T}^\star V'$. It is a well-known fact that $\mathrm{T}^\star_C V$ is a lagrangian conical variety.

Note that since f is quasihomogeneous and non-degenerate, it is easy to calculate the fibres of $\mathrm{T}^\star_C V$ everywhere, except at the origin. However, since $\mathrm{T}^\star_C V$ is irreducible and lagrangian, it follows that it cannot contain the whole fibre at the origin, denoted by $\mathrm{T}^\star_0 V$. This will eventually lead to the contradiction that will allow us to prove the theorem of §1.

Consider $\mathcal{O}_V[f^{-1}]$; it is naturally a coherent $\mathcal{D}_V$-module. We will show that the characteristic variety of this module contains the fibre $\mathrm{T}^\star_0 V$. To do this we will make use of a description of the characteristic variety due to Kashiwara. A more general result is proved in [LM] but we give a simpler approach to the special case we need.

2.1 Proposition. *The characteristic variety of* $\mathcal{O}_V[f^{-1}]$ *is*

$$\{(v, \theta) \in \mathrm{T}^\star V : \theta \wedge df(v) = 0 \quad \textit{and} \quad f(v)\theta = 0\}.$$

In particular, it contains $\mathrm{T}^\star_0 V$*, the fibre at the origin of* $\mathrm{T}^\star V$*.*

Proof. Define

$$W' = \{(v, \theta) \in \mathrm{T}^\star V : \theta = s(df/f)(v), \ \ f(v) \neq 0 \ \text{ and } \ \ s \in \mathbb{C}^\times\},$$

and let W be the (Zariski) closure of W' in $\mathrm{T}^\star V$.

It is is proved in [SKKO, Appendix] after [K, Proposition 6.2] (see also [Gi, Theorem 3.3], [Gy, §2.4]) that the characteristic variety of $\mathcal{O}_V[f^{-1}]$ is equal to

$$\{(v, \theta) \in W : f(v)\theta = 0\}.$$

Let $y_1, \ldots, y_n$ be a basis of V and $x_1, \ldots, x_n$ the dual basis of $V^\star$. Write $S = S(V^\star) = \mathbb{C}[x_1, \ldots, x_n]$ and write J for the Jacobian ideal $\sum_i \partial f/\partial x_i S$. Let

$$\widetilde{W} = \{(v, \theta) \in \mathrm{T}^\star V : \theta \wedge df(v) = 0\}.$$

Now $\widetilde{W} = \mathcal{Z}(I)$, where I is the ideal of $S(V^\star \times V) = S[y_1, \ldots, y_n]$ generated by

$$y_j \partial f/\partial x_i - y_i \partial f/\partial x_j.$$

Since $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$ is a regular sequence, the map $S[y_1, \ldots, y_n]/I \to S_S(J)$ defined by $y_i + I \mapsto \partial f/\partial x_i$ is an isomorphism. But the symmetric algebra is a domain by [Ei, Ex. 17.14]. Hence $\widetilde{W}$ is a closed irreducible subset of $\mathrm{T}^\star V$.

Note that $(v,\theta) \in \widetilde{W}$ if and only if $df(v)$ and θ are linearly dependent. As f is quasihomogeneous and non-degenerate, we have $df(v) = 0$ if and only if $v = 0$. Thus,

$$\widetilde{W} = \{(v,\theta) \in \mathrm{T}^\star V : \theta = s\,df(v), \text{ for some } s \in \mathbb{C}, \text{ or } v = 0\}.$$

It follows that

$$W' = \widetilde{W} \cap \{(v,\theta) : f(v)\theta \neq 0\}.$$

Therefore the Zariski closure of W' must be contained in $\widetilde{W}$; since the latter is irreducible, $W = \widetilde{W}$. It is now clear that $\mathrm{Ch}(\mathcal{O}_V[f^{-1}])$ is as in the statement and contains $\mathrm{T}_0^\star V$. □

The module that plays the crucial rôle in the proof of the theorem is not $\mathcal{O}_V[f^{-1}]$ itself but

$$\mathcal{H} = \mathcal{O}_V[f^{-1}]/\mathcal{O}_V. \tag{2.2}$$

This $\mathcal{D}_V$-module is isomorphic to $\mathcal{H}^1_{[C]}(\mathcal{O}_V)$, the first local cohomology sheaf with support in C. It is a regular holonomic $\mathcal{D}_V$-module, by [BK, Proposition 1.3(7)]. We next show that $\mathrm{Ch}(\mathcal{H})$ contains $\mathrm{T}_0^\star V$.

2.3 Proposition. $\mathrm{Ch}(\mathcal{H}) = \mathrm{T}_C^\star V \cup \mathrm{T}_0^\star V$.

Proof. Let $U = \mathrm{T}^\star V \setminus \mathrm{T}_0^\star V$. Note that $U \cap \mathrm{Ch}(\mathcal{H}) = \mathrm{Ch}(\mathcal{H}|_{V'}) = \mathrm{T}_{C'}^\star V'$. It follows at once that $\mathrm{Ch}(\mathcal{H}) \supseteq \mathrm{T}_C^\star V$. Further, if there exists another irreducible component of $\mathrm{Ch}(\mathcal{H})$, then it contains $\mathrm{T}_0^\star V$.

Recall that $\mathcal{H} \cong \mathcal{H}^1_{[C]}(\mathcal{O}_V)$. Now, (2.2) implies that

$$\mathrm{Ch}(\mathcal{O}_V[f^{-1}]) = \mathrm{Ch}(\mathcal{O}_V) \cup \mathrm{Ch}(\mathcal{H}).$$

By Proposition 2.1, the fibre $\mathrm{T}_0^\star V$ is contained in the left-hand side, but not in the characteristic variety of $\mathcal{O}_V$, which is the zero section of $\mathrm{T}^\star V$. Hence $\mathrm{Ch}(\mathcal{H})$ contains $\mathrm{T}_0^\star V$, and the proposition is proved. □

Recall that the *link* of $\mathcal{Z}(f)$ is the intersection of C with a small sphere centred at the origin. Recall also the definition of the *Bernstein polynomial* [Co, p. 94]. This is the polynomial $b(s)$ in an indeterminate s which has least degree amongst polynomials for which there exists a differential operator p with the property that $pf^{s+1} = b(s)f^s$. From the definition, it is clear that $b(s) = (s+1)\tilde{b}(s)$.

2.4 Proposition. *The following are equivalent:*

(1) *$\mathcal{H}$ is simple;*
(2) *the link of $\mathcal{Z}(f)$ is a $\mathbb{Q}$-homology sphere;*
(3) *$\tilde{b}(s)$ has no integer roots.*

Proof. Let $i : C' \to V'$ be the inclusion. Then $\mathcal{H}|_{V'} \cong \mathcal{H}^1_{[C']}(\mathcal{O}_{V'})$ is simple since, by Kashiwara's equivalence (see [Bo, Theorem VI.7.13(ii), Theorem VI.7.4(i)(ii)(iii)]), one has that $i_+(\mathcal{O}_{C'}) \cong \mathcal{H}^1_{[C']}(\mathcal{O}_{V'})$. It follows from this (see [BK, Proposition 8.5]) that $\mathcal{H}$ has a simple socle $\mathcal{L}$ and that $\mathcal{H}/\mathcal{L} \cong \mathcal{S}^t$ is a direct sum of t copies of the simple module $\mathcal{S}$ supported only at the origin. Now consider the Riemann-Hilbert solution functor $S = R\mathcal{H}\mathrm{om}_{\mathcal{D}_V}(\_, \mathcal{O}_V)$ which gives a (contravariant) equivalence between the category of regular holonomic $\mathcal{D}_V$-modules and the category of perverse sheaves on V [Bo, Theorem VIII.14.4]. One has $S(\mathcal{H}) = \mathbb{C}_C[-1]$, where $\mathbb{C}_C$ is extended by zero to a sheaf on V, and likewise $S(\mathcal{S}) = \mathbb{C}_{\{0\}}[-n]$. Further, by [BK, Theorem 8.6], one has $S(\mathcal{L}) = \mathcal{IC}_C[-1]$, the (shifted) intersection cohomology complex [GM]. Thus, one has a short exact sequence:

$$0 \to \mathbb{C}_{\{0\}}[-n]^t \to \mathbb{C}_C[-1] \to \mathcal{IC}_C[-1] \to 0.$$

Since $\mathcal{K} = \mathbb{C}_{\{0\}}[-n]^t$ is concentrated in degree n, and $\mathbb{C}_C[-1]$ is concentrated in degree 1, we see that in the bounded derived category $\mathcal{K} \cong \mathcal{H}^{n-2}(\mathcal{IC}_C)$. On the other hand, using the Whitney stratification $C = C' \cup \{0\}$ and Deligne's approach to $\mathcal{IC}$ [GM, §3], we get

$$\mathcal{IC}_C \cong \tau_{\leq n-2} Rj_* \mathbb{C}_{C'},$$

where $j : V' \to V$ is the inclusion and τ is truncation. It follows that $\mathcal{K}$ is the skyscraper sheaf at the origin with stalk

$$\lim_{\epsilon \to 0} H^{n-2}(C' \cap B_\epsilon, \mathbb{C}).$$

This limit is $H^{n-2}(K, \mathbb{C})$, where K is the link of C.

It remains to remark that K is a $\mathbb{Q}$-homology sphere if and only if $H^{n-2}(K, \mathbb{C}) = 0$. But K is a compact oriented $(2n-3)$-manifold which is $(n-3)$-connected, by [Mi, Theorem 5.2], and so by the Hurewicz theorem and Poincaré duality we have

$$H_0(K, \mathbb{C}) \cong H_{2n-3}(K, \mathbb{C})^* \cong \mathbb{C} \qquad \text{and} \qquad H_{n-2}(K, \mathbb{C}) \cong H_{n-1}(K, \mathbb{C})^*,$$

with all other homology groups equal to zero. The equivalence of (1) and (2) follows.

Finally, let F denote the Milnor fibre, $F = \mathcal{Z}(f-1)$. The monodromy $F \to F$, given by $v \mapsto e^{2\pi i/r} v$, induces an operator on $H_{n-1}(F, \mathbb{C})$. Further, by [Di, Proposition 3.4.7], K is a $\mathbb{Q}$-homology sphere if and only if 1 is not an eigenvalue for the monodromy operator. However, by [Ma, Théorème 5.4], 1 is an eigenvalue of the monodromy operator if and only if $\tilde{b}$ has an integer root. □

3. Proof of theorem 1.1

Suppose that there exists a $\mathcal{D}_V$-module $\mathcal{M}$ such that $\mathrm{Ch}(\mathcal{M}) = \mathrm{T}^\star_C V$ and let us aim at a contradiction. Since $\mathrm{T}^\star_C V$ is irreducible, we may as well assume that $\mathcal{M}$ is simple. Note that C' is homotopy equivalent to the link of C and so C' is simply connected.

It follows from Kashiwara's equivalence ([Bo, Theorem VI.7.11]) that $\mathcal{M}|_{V'} = i_+(\mathcal{N})$, for some simple $\mathcal{D}_{C'}$-module $\mathcal{N}$ with characteristic variety $\mathrm{T}^\star_{C'} C'$. Since C' is simply connected, $\mathcal{N} \cong \mathcal{O}_{C'}$. Thus, $\mathcal{M}|_{V'} \cong \mathcal{H}^1_{[C']}(\mathcal{O}_{V'})$.

Now, looking at the characteristic variety, the support of $\mathcal{M}$ is not concentrated at the origin. Thus, if $j : V' \to V$ is the canonical embedding, there is a monomorphism $k : \mathcal{M} \to \mathcal{G} := j_*(\mathcal{H}|_{V'})$. Of course, $k(\mathcal{M})|_{V'} = \mathcal{G}|_{V'}$. Likewise, $\mathcal{H}$ is a submodule of $\mathcal{G}$ with $\mathcal{H}|_{V'} = \mathcal{G}|_{V'}$. Since the quotient $\mathcal{G}/k(\mathcal{M})$ can only possibly be supported at the origin, we cannot have $\mathcal{H} \cap k(\mathcal{M}) = 0$ and so $\mathcal{H} = k(\mathcal{M}) \cong \mathcal{M}$.

Of course, as $\mathrm{T}^\star_C V$ is irreducible, $\mathrm{Ch}(\mathcal{M})$ doesn't contain the zero fibre $\mathrm{T}^\star_0 V$. On the other hand, by Proposition 2.3, we have that $\mathrm{Ch}(\mathcal{H}) \supseteq \mathrm{T}^\star_0 V$. This contradiction completes the proof of the theorem.

4. Examples

It is now easy to give many examples: let $f = x_1^{a_1} + \cdots + x_n^{a_n}$ with $n \geq 4$, and each $a_i \geq 2$. Consider the graph G with vertices $1, \ldots, n$ and an edge linking i and j if and only if $hcf(a_i, a_j) > 1$. Let G_{ev} denote the component of G containing all the i with a_i even. Suppose that either (a) G contains an isolated point, or (b) G_{ev} contains an odd number of vertices and $hcf(a_i, a_j) = 2$, for all distinct i, j in G_{ev}. After Brieskorn [Br], f satisfies the hypotheses of the theorem. Note that the example in the introduction is a special case of this one.

We complete the paper by showing that some conormal varieties of quasihomogeneous non-degenerate hypersurfaces are characteristic varieties.

Example. Let $f = \sum_{i=1}^n x_i^2$ and suppose that $n \geq 4$. Then $T_C^\star V$ is a characteristic variety if and only if n is even.

Proof. The case when n is odd is covered by Theorem 1.1 and the previous example. So suppose that n is even and let $p = \sum_{i=1}^n (\partial/\partial x_i)^2$. Then,

$$pf^{s+1} = 4(s+1)(s+\frac{n}{2})f^s.$$

It follows that if $s = -\frac{n}{2}$, then p annihilates f^{s+1}. Thus, the symbol of p vanishes on $\mathrm{Ch}(\mathcal{G})$ where $\mathcal{G}$ is the submodule $(\mathcal{D}f^{s+1} + \mathcal{O})/\mathcal{O}$ of $\mathcal{H}$. Now, $\mathrm{Ch}(\mathcal{G}) \subseteq \mathrm{Ch}(\mathcal{H})$ and hence either $\mathrm{Ch}(\mathcal{G}) = \mathrm{T}_C^\star V$ or $\mathrm{Ch}(\mathcal{G}) = \mathrm{T}_C^\star V \cup \mathrm{T}_0^\star V$. Since the symbol of p doesn't vanish on $\mathrm{T}_0^\star V$ we see that the second possibility does not occur. □

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