NON-HOLONOMIC SIMPLE $\mathcal{D}$-MODULES OVER COMPLETE INTERSECTIONS

S. C. COUTINHO

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Abstract. We show that if X is a complex affine algebraic variety whose projective closure is a smooth complete intersection of dimension $n \geq 3$ then there exist non-holonomic simple $\mathcal{D}(X)$-modules of dimension $n + 1$.

Let X be an irreducible affine smooth algebraic variety of dimension n over $\mathbb{C}$. A module M over the ring of differential operators $\mathcal{D}(X)$ is *holonomic* if its Gelfand-Kirillov dimension is equal to n. Simple holonomic modules are easy to produce on industrial scale for any given X. By contrast, although J. Bernstein and V. Lunts have shown in [1] that most simple modules over the Weyl algebra $\mathcal{D}(\mathbb{C}^n)$ are non-holonomic, very few examples are known when $X \neq \mathbb{C}^n$. Some of these can be found in [3, section 4, p. 413].

In this note we construct non-holonomic simple $\mathcal{D}(X)$-modules when X is a general complete intersection. The main result is as follows.

Theorem 1. *Let X be an affine algebraic closed subvariety of $\mathbb{A}^N(\mathbb{C})$ whose projective closure $\overline{X}$ is an irreducible smooth complete intersection. If $\dim(X) > 2$ then $\mathcal{D}(X)$ admits a simple module of Gelfand-Kirillov dimension $n+1$.*

One of the key ingredients of the proof is a result of L. G. Mendes on holomorphic foliations without algebraic solutions. Let S be a smooth complex projective surface. A 1-*dimensional singular foliation* over S is a rank one locally free $\mathcal{O}_S$-submodule of the tangent sheaf Θ_S. Since a foliation is locally determined by a derivation of S, we may define what it means for a curve to be invariant under a foliation in terms of derivations. If d is a derivation of an affine variety X then an algebraic set $Y \subseteq X$ is *invariant* under d if its ideal $I(Y)$ in $\mathcal{O}(X)$ satisfies $d(I(Y)) \subseteq I(Y)$.

Mendes's theorem makes use of the pull-back of a holomorphic foliation, which is defined in [11, p. 183]. For a proof of this theorem see [9, theorem 1].

Theorem 2. *Let $\pi : S \to \mathbb{P}^2$ be a finite projection and let $\mathcal{F}$ be a foliation of $\mathbb{P}^2$ with no invariant algebraic curves. The pull-back $\pi^\dagger(\mathcal{F})$ is a foliation of S with the same property. Moreover, the singularity sets of the two foliations are related by $\pi^{-1}(\mathrm{Sing}(\mathcal{F})) \subset \mathrm{Sing}(\pi^\dagger(\mathcal{F}))$.*

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Another key ingredient is a generalization of a result of J. Bernstein and V. Lunts. In [1, section 4] simple modules over the second Weyl algebra are constructed from a derivation with no invariant algebraic set, except its singularities. This result generalizes to smooth surfaces as the next theorem shows. The proof is a straightforward generalization of the argument in [1, section 4] and will be omitted.

Theorem 3. *Let W be a smooth affine algebraic surface over $\mathbb{C}$ whose Picard group is trivial. If d is a derivation of W with no invariant algebraic curves and a finite non-empty singularity set then there exists $f \in \mathcal{O}(W)$ such that $\mathcal{D}(W)/\mathcal{D}(W)(d+f)$ is a non-holonomic simple $\mathcal{D}(W)$-module.*

Recall that hypersurfaces of degree $k \geq 1$ in $\mathbb{P}^N$ are parametrized by the space $\mathbb{P}^{\nu_{N,k}}$, where $\nu_{N,k} = \binom{N+k}{k} - 1$. A property $\mathcal{P}$ is said to hold for a *generic* hypersurface if the set of points $p \in \mathbb{P}^{\nu_{N,k}}$ such that the hypersurface corresponding to p does not satisfy $\mathcal{P}$ is contained in a countable number of algebraic hypersurfaces of $\mathbb{P}^{\nu_{N,k}}$. The following lemma is an immediate consequence of Bertini's theorem.

Lemma 4. *Let V be a smooth irreducible algebraic variety in $\mathbb{P}^N$. If Y is a generic hypersurface of $\mathbb{P}^N$, then the variety $V \cap Y$ is irreducible and smooth.*

Proposition 5. *Let V be an irreducible smooth complete intersection in $\mathbb{P}^N$. If $\dim(V) > 2$ then there exists a smooth surface S contained in V whose Picard group is the free abelian group generated by $H \cdot S$, where H is the hyperplane class of $\mathbb{P}^N$.*

Proof. Suppose first that $\dim(V) = d > 3$, and let Z be a generic hypersurface of $\mathbb{P}^N$. It follows from lemma 4 that $W_{d-1} = V \cap Z$ is a smooth irreducible variety of dimension $d - 1 > 2$. By [6, theorem 17.18, p. 218] W_{d-1} is also a complete intersection. By Lefschetz's theorem on complete intersections we have that $\mathrm{Pic}(W_{d-1})$ is the free abelian group generated by $W_{d-1} \cdot H$, where H is the hyperplane class of $\mathbb{P}^N$; see [10, theorem 8.5, p.248] or [5, Exposé XII Corollaire 3.7, p. 153].

Proceeding by induction, we conclude that V contains an irreducible smooth complete intersection W_3 of dimension 3, whose Picard group is the free abelian group generated by $W_3 \cdot H$. Now it follows from [10, theorem 7.5, p.247] that there exists a positive integer m_0 such that, for any generic hypersurface Z of $\mathbb{P}^N$ of degree $m \geq m_0$, the Picard group of the intersection $W_2 = W_3 \cap Z$ is the free abelian group with basis $H \cdot W_2$. As before, W_2 is an irreducible, smooth, complete intersection in $\mathbb{P}^N$.

Proof of theorem 1. Choose homogeneous coordinates $[x_0 : \ldots, : x_N]$ for $\mathbb{P}^N$ such that $X \subset U_0$, where U_i is the open set of $\mathbb{P}^N$ defined by $x_i \neq 0$. By theorem 5 there exists a smooth surface $S \subseteq \overline{X}$ for which $\mathrm{Pic}(S)$ is the free abelian group generated by $H \cdot S$, where H is the class of the hyperplane section. Now, choose a foliation $\mathcal{G}$ of $\mathbb{P}^2$ which has no invariant algebraic curve, and such that $\pi^{-1}(\mathrm{Sing}(\mathcal{G})) \cap U_0 \neq \emptyset$. The existence of holomorphic foliations with these properties follows from a famous result of Jouanolou; see [8, p. 157ff]. It follows from theorem 2 that S admits a foliation $\mathcal{F}$ with no invariant algebraic curves and such that $\mathrm{Sing}(\mathcal{F}) \cap U_0 \neq \emptyset$.

Let $S_0 = S \cap U_0$. Then S_0 is a smooth surface contained in X, and by [7, Proposition II.6.5(c), p. 133] $\mathrm{Pic}(S_0) = 0$. Since $\mathcal{F}$ is a locally free sheaf over S, it follows that $\mathcal{F}(S_0)$ is a projective $\mathcal{O}(S_0)$-module of rank 1. Thus $\mathcal{F}(S_0)$ is cyclic, generated by a derivation d of S_0. Since $\mathcal{F}$ has no invariant curves, and $\mathrm{Sing}(\mathcal{F}) \cap U_0 \neq \emptyset$, it folllows that

(1) d has no invariant curves;
(2) d has singularities in S_0.

Now, by theorem 3 there exists $f \in \mathcal{O}(S_0)$ such that $M_0 = \mathcal{D}(S_0)/\mathcal{D}(S_0)(d+f)$ is a simple $\mathcal{D}(S_0)$-module of codimension 1. Since S_0 is a surface, M_0 has holonomic defect 1.

Let $i : S_0 \to X$ be the natural embedding of S_0 into X, and denote by i_* the corresponding direct image functor of $\mathcal{D}$-modules. By Kashiwara's equivalence $i_*(M_0)$ is a simple $\mathcal{D}(X)$-module; see [2, theorem 7.11, p.263]. Since holonomic defect is preserved by direct image under closed embeddings, it follows that $i_*(M_0)$ has dimension $n+1$, and the theorem is proved.

The proof of theorem 1 breaks down for surfaces because they need not have a trivial Picard group. Indeed, in dimension 2, the analogue of theorem 1 holds only for generic surfaces.

Theorem 6. *Let X be an algebraic surface in $\mathbb{A}^N(\mathbb{C})$. If $\overline{X}$ is a generic complete intersection of degree greater than or equal to* 4 *then there exists a derivation d and a regular function f of X such that $\mathcal{D}(X)/\mathcal{D}(X)(d+f)$ is a simple $\mathcal{D}(X)$-module of dimension* 3.

Proof. By theorem 2 the surface $\overline{X}$ admits a foliation $\mathcal{F}$ with no invariant curves and with at least one singular point in X.

Since $\overline{X}$ is a generic complete intersection of degree at least 4, it follows from [4, théorème 1.2, p. 328] or [10, theorem 8.5, p. 248] that $\mathrm{Pic}(\overline{X})$ is the free abelian group with basis $H \cdot \overline{X}$, where H is the class of the hyperplane divisor in $\mathbb{P}^N$. Hence $\mathrm{Pic}(X) = 0$. Thus $\mathcal{F}(X)$ is cyclic generated by a derivation d. The result now follows by theorem 3.

Note that the easiest way to extend these results to all smooth irreducible algebraic varieties is to settle the following problem in the affirmative.

Problem 7. *Let X be a smooth irreducible complex affine variety of dimension greater than* 2. *Does X contain a smooth surface S whose Picard group is trivial?*

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Departamento de Ciências da Computação, Instituto de Matemática, Universidade Federal do Rio de Janeiro, P.O. Box 68530, 21945-970 Rio de Janeiro, RJ, Brazil.
E-mail address: collier@impa.br